The Comprehensive Examination, Part A - Algebra -- for

Graduate Students in the Department of Pure

Mathematics

10 a.m.-1 p.m. October 27, 1978

Do as many questions as possible

1. a) Determine the structure of the Galois group of the splitting field of $x^4-2 = 0$ (over the rational numbers Q).

   b) Let ζ be a primitive n'th root of unity $(n \geq 3)$ and let $\theta = \zeta+\zeta^{-1}$. Show that $[Q(\theta): Q] = \frac{1}{2}\phi(n)$, where ϕ is the Euler function. You may use the fact that $[Q(\zeta): Q] = \phi(n)$.

2. a) Let R be a principal ideal domain. Suppose that $0 \neq x \in R$ is an element such that $x = yz$ implies y is a unit or z is a unit. Show that if $x|ab$, then $x|a$ or $x|b$. (elf means 'e divides f'.)

   b) Show that a group of order 12 which has no elements of order 6 is isomorphic to A_4 (the alternating group).

3. a) Define: noetherian module, artinian module. Prove that if a module M is artinian and noetherian, then there is a finite chain of submodules

$$M = M_1 \supset M_2 \supset \dots \supset M_k = (0)$$

   such that M_i/M_{i+1} is irreducible for each i.

   b) Let M be a finitely generated (left) R-module, and let J be the Jacobson radical of R. Show that if $JM = M$, then $M = (0)$.

   c) State the Hilbert Basis Theorem.

4. a) Show that an abelian group is divisible if and only if it has no maximum subgroup.

4. (con'd)

   b) Give an example of a finitely generated group with a subgroup that is not finitely generated.

   c) Show that if G is a finitely generated group and H is a proper subgroup, then there is a maximal subgroup of G which contains H.

5. a) Show that there is only one projective plane Π of order 2 (up to isomorphism). This means that every line on Π consists of 3 points.

   b) Compute the order of the automorphism group of Π.