UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

The Comprehensive Examination in Algebra

Department of Pure Mathematics, April 30, 1979

TIME: 3 hours plus.

1. (a) State the Sylow Theorems. Prove that the centre of a group G cannot be of index 77 in G.

   (b) Let $\sigma, \rho: \mathbb{R} \to \mathbb{R}$ be defined by $\sigma(x) = x+1$, $\rho(x) = 2x$. Show that $G = \langle\sigma, \rho\rangle$ is a solvable subgroup of the symmetric group on $\mathbb{R}$. Also show that G has a subgroup which is not finitely generated. (Hint: let $\sigma_n = \rho^{-n}\sigma\rho^n$ and show that $\sigma_n{}^2 = \sigma_{n-1}$. Note that we are defining the group operation by $(\alpha\beta)(x) = \alpha(\beta(x))$.)

   (c) Let G be a torsion-free group. Show that G can be embedded in a simple group. (Hint: use a suitable theorem of Higman, Neumann, and Neumann repeatedly.)

2. Indicate briefly how you would prove that if R is a unique factorization domain then so is $R[x]$.
   Determine whether or not $R = \mathbb{Q}[x, x^{-1}]$ is a unique factorization domain. (Note: R is the subring of $\mathbb{Q}(x)$ which consists of polynomials over $\mathbb{Q}$ in x and x^{-1}.)

3. Let $f(x) = x^5-4x+2$.

   (i) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

   (ii) Show that $f(x)$ has precisely three real roots.

   (iii) Find the Galois group G of $f(x)$ over $\mathbb{Q}$.

   (Hint: interpret G as a group of permutations

3. (con'd)

on the roots of $f(x)$ and show that it contains a transposition.)

(iv) If E is the splitting field of $f(x)$ over $\mathbb{Q}$, find all subfields of E which are Galois over $\mathbb{Q}$.

4. a) Let A be a 10×10 matrix over $\mathbb{C}$ with characteristic polynomial $(x-4)^5(x-2)^3(x-3)^2$ and minimal polynomial $(x-4)^3(x-2)^2(x-3)$. Suppose that the dimension of the eigenspace corresponding to the eigenvalue 4 is 2.

(i) Find a matrix in Jordan form which is similar to A.

(ii) What are the dimensions of the eigenspaces corresponding to the other eigenvalues of A?

b) For which of the following matrices A does there exist a unitary matrix P such that P^*AP is diagonal?

(i) $A = \begin{pmatrix} 1 & i & -2 \\ i & 2 & 1 \\ -2 & 1 & i \end{pmatrix}$ (ii) $A = \begin{pmatrix} -1 & -8 & -4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{pmatrix}$.

5. Prove that every abelian group is isomorphic to a subgroup of a divisible abelian group.
State the analogous result for modules, defining the term(s) used. Describe a divisible abelian group G such that the cyclic group of order p is isomorphic to a subgroup of G.

6. Prove that the intersection $\mathrm{rad}\, R$ of all the prime ideals of a commutative ring R consists precisely of the nilpotent elements of R. Compute $\mathrm{rad}\, R$ in the cases $R = \mathbb{Z}_{360}$, $R = \mathbb{Z}_4[x]/(x^2)$.

7. Give examples of modules A_R which are

(i) Artinian and Noetherian,

(ii) Artinian but not Noetherian,

(iii) Noetherian but not Artinian,

(iv) Neither Artinian nor Noetherian.

Note: It is preferred that the A_R's used be of the form R_R; if this is not possible, give reasons and give a suitable A_R.

8. EITHER

(I) State the minimax formulae (given in terms of "Rayleigh quotients") for the eigenvalues of a Hermitian matrix A. Deduce a relationship between the eigenvalues of A and of the matrix B obtained by omitting the last row and column of A.

OR

(II) Discuss various methods for calculating the eigenvalues of a real symmetric matrix A.