UNIVERSITY OF WATERLOO

WATERLOO ONTARIO

The Comprehensive Examination in Algebra

Department of Pure Mathematics

October 31, 1979

TIME: 3 Hours

Answer questions 1,2,3 and 3 further questions.

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1. (a) State a version of the Sylow Theorems.

   (b) Show that a group of order pq, where p and q are primes, cannot be simple. Determine when such a group is necessarily abelian.

   (c) Let G be the group of invertible $3x3$ matrices over the field $\mathbf{F}_p$ of p elements, p prime. Determine the order of G and hence or otherwise find a Sylow p-subgroup of G.

2. (a) Let R be a commutative ring with 1. Show that every maximal ideal of R is prime and that the converse holds if R is finite.

   (b) Prove that every principal ideal domain is a unique factorization domain.

   (c) Show that $\mathbb{Z}[i]$ is a unique factorization domain.

3. Let $f(X) = X^p - a$, where p is prime, be irreducible in $\mathbb{Q}[X]$. Prove that the Galois group of $f(X)$ over $\mathbb{Q}$ is isomorphic to the group of transformations of $\mathbb{Z}/(p)$ of the form $z \mapsto kz + \ell$ where $k,\ell \in \mathbb{Z}/(p)$ and $k \neq 0$.

4. Let V be a left vector space over the skewfield K and W a left vector space over the skewfield F. Suppose that dimension of V over K is 3 and the dimension of W over F is 3. Suppose also that there is an isomorphism of the projective plane $P(V)$ onto the projective plane $P(W)$. Prove that F and K are isomorphic and that the isomorphism from $P(V)$ onto $P(W)$ is induced by a semilinear transformation from V onto W.

5. (a) Suppose α is a (A,ℓ)-perspectivity and β is a (B,ℓ)-perspectivity. Suppose also that α and β are non-trivial and that $A \neq B$. Prove that $\alpha\beta$ is a non-trivial (C,ℓ)-perspectivity where $C \neq A$ and $C \neq B$.

   (b) Let P be a projective plane and let Γ be any collineation group of P. Suppose that $\Gamma_{(P,\ell)}$ is non-trivial for two distinct choices of P on ℓ.
   Prove that $\Gamma_{(\ell,\ell)}$ is abelian and all its non-identity elements have the same order (either a prime or infinite).

6. A subgroup H of a group G is said to be a <u>fully invariant</u> subgroup of G if every endomorphism of G maps H into itself.

   (a) Let $N \triangleleft G$ and let H be a fully invariant subgroup of N. Show that $H \triangleleft G$.

   (b) Show that the commutator subgroup G^1 of G is fully invariant.

   (c) Show that if A is an abelian group of order p^α, p prime, then the set of elements of order $\leq p$ in A is a fully invariant subgroup of A.

   (d) Show that if H is a minimal (non-trivial) normal subgroup of a finite solvable group G then H is elementary abelian.

7. (a) Define 'simple ring'. Prove that the endomorphism ring R of a finite-dimensional vector space V over a division ring D is simple. Discuss the converse (wherein 'is' is to be replaced by 'is isomorphic to'). (If the converse is not invariably true, give a counterexample and describe additional conditions under which it is true.)

   (b) Define 'von Neumann regular ring'. Prove that a right Noetherian von Neumann regular ring is right Artinian.

8. Calculate the cardinalities of the automorphism groups of the following fields: $\mathbb{R}$, $\mathbb{C}$, F_q - the finite field of $q = p^n$ elements, p prime.

9. (a) Find the Jordan canonical form of

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

   Hence show that A is similar to its transpose.

   (b) Show that if V is an n-dimensional vector space over $\mathbb{Q}$ then there exists a linear map $A: V \to V$ such that $v, A(v), \ldots, A^{n-1}(v)$ are linearly independent for every non-zero v in V. Would such an A be possible if we replaced $\mathbb{Q}$ by $\mathbb{C}$?