PURE MATHEMATICS

ALGEBRA

COMPREHENSIVE EXAMINATION

TIME: 3 HOURS Fall, 1991

Answer **six** questions and include one from each of the topics: Group Theory, Linear Algebra, Ring Theory, and Field Theory.
Be sure to justify each of your answers.

Linear Algebra

1. Let T be a normal operator on a finite-dimensional complex inner product space V.

   (a) Show that $\|Tu\| = \|T^*u\|$ for all $u \in V$.

   (b) If $T^2u = 0$ show that $Tu = 0$.

   (c) Prove that the eigenvalues of T are all equal if and only if $T = cI$ for some number c, where I is the identity transformation.

2. (a) Let A be a real symmetric $n \times n$ matrix with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

   For any non-zero vector $x \in \mathbb{R}^n$, show that

$$\lambda_1 \leq \frac{\langle x, Ax\rangle}{\langle x, x\rangle} \leq \lambda_n.$$

   (b) Find a matrix A with 1 as the largest modulus of its eigenvalues and such that

$$\frac{\langle x, Ax\rangle}{\langle x, x\rangle} > 1$$

   for some non-zero vector x.

(c) If $\{w_1, \ldots, w_m\}$ is an orthonormal set in a finite-dimensional inner product space V such that

$$\sum_{i=1}^{m} |\langle w_i, v\rangle|^2 = \|v\|^2 \quad \text{for every } v \in V$$

prove that $\{w_1, \ldots, w_m\}$ is a basis of V.

Group Theory

3. (a) If a finite group G acts on a set X and $x \in X$, prove that the number of elements in the orbit of x is the index of the stabilizer of x in G.

   (b) The general linear group, $GL_2(\mathbb{Z}_p)$, of invertible 2×2 matrices over the finite field $\mathbb{Z}_p$, acts on all the 2×2 matrices over $\mathbb{Z}_p$ by left multiplication of matrices. Determine the order of the stabilizer subgroup of each 2×2 matrix.

4. (a) State the first Sylow Theorem.

   (b) If G is a group of order pq where p and q are primes, prove that either G is cyclic or G is generated by two elements a and b satisfying

$$b^p = 1; \quad a^q = 1; \quad a^{-1}ba = b^r.$$

Ring Theory (NOTE: All rings are to be considered as rings with identity. All modules are left modules such that $1 \cdot m = m$ forall m.)

5. (a) Prove principal ideal domains are unique factorization domains.

   (b) Which $\mathbb{Z}_n$ are Euclidean domains? Principal ideal domains? Unique factorization domains?

   (c) Show that $\mathbb{Q}[\sqrt{-1}]$ is a Euclidean domain.

6. (a) Let R be a finite commutative ring and let $J(R)$ be the Jacobson radical of R. Describe $R/J(R)$. Show that there is an integer $n \geq 2$ so that every element of this quotient ring satisfies $x^n = x$.

   (b) Let R be the subring of $\mathbb{C}^N$, where N is the set of natural numbers and $\mathbb{C}$ is the ring of complex numbers, given by

$$R = \{f \in \mathbb{C}^N : \text{Range}(f) \text{ is finite}\}.$$

   (i) Verify that R is indeed a subring of $\mathbb{C}^N$.

   (ii) Calculate the Jacobson radical of R.

Field Theory

7. (a) Outline the key ideas in the proof that for every prime p and positive integer n there is, up to isomorphism, exactly one field of size p^n.

   (b) Find the monic polynomial of smallest degree in $\mathbb{Z}_2[x]$ which vanishes on both $\mathbf{GF}(4)$ and $\mathbf{GF}(8)$.

8. (a) Prove or disprove: the number $a \in \mathbb{C}$ satisfies $[\mathbb{Q}(a) : \mathbb{Q}]$ is a power of two implies it is constructible (by straight edge and compass), given 0 and 1.

   (b) Prove that the automorphism group of the algebraic closure of $\mathbb{Q}$ has the size c of the continuum.

   (c) Prove the automorphism group of the complex field $\mathbb{C}$ has size 2^c, c being the size of the continuum.