Algebra Comprehensive

Fall 1990

Complete as much as you can, but be sure to do several parts [(a), (b),...] from each of the four questions.

1. (a) (i) State the structure theorem for finitely-generated abelian groups.
   (ii) Give the strongest generalization of (i) to modules that you know.

   (b) (i) Define: divisible abelian group.
   (ii) Relate the divisibility of $G \oplus H$ to that of G and of H.
   (iii) Find all finitely-generated divisible abelian groups.
   (iv) Show that if A is a divisible abelian group then for all integers n and all group morphisms $f : n\mathbb{Z} \to A$ there exists a morphism $g : \mathbb{Z} \to A$ which restricts to f. Is the converse true?

   (c) (i) Give an example without proof of a finitely-generated group G having a normal subgroup H which is not finitely-generated.
   (ii) If N is a finitely-generated normal subgroup of G and G/N is finitely-generated, is it necessarily true that G is finitely-generated?

   (d) (i) What is $\mathrm{N}_G(H)$, the normalizer of H in G?
   (ii) Show that $[G : N_G(H)]$ is the number of subgroups of G conjugate to H. (Note: it is not assumed that G is finite.)

   (e) (i) State the first Sylow theorem.
   (ii) What is a nilpotent group?
   (iii) Show that a group is nilpotent iff it is the direct product of its Sylow subgroups.
   (iv) Write S_4 as a direct product of its Sylow subgroups or explain why it is not possible to do so.

(f) Suppose $D_m \approx S_n$, where D_m is the dihedral group of order $2m$ and S_n is the symmetric group on n letters. Show that $m = 1$ and $n = 2$, or $m = n = 3$.

(g) Find, with explanation but not a complete proof, all pairs (p, q) of primes for which there exists a finite simple group whose order is divisible by pq and by no primes other than p and q.

2. (a) (i) Give an example of an integral domain which is **not** a unique factorization domain.

   (ii) Give an example of a unique factorization domain which is **not** a principal ideal domain.

   (iii) Give an example of a single $g(x) \in \mathbb{Q}[x]$ whose factorization into irreducibles is different in each of $\mathbb{Q}[x], \mathbb{Q}(\sqrt{2})[x], \mathbb{R}[x]$ and $\mathbb{C}[x]$.

   (b) State the Hilbert basis theorem.

   (c) Define the term **semi-simple ring.**

   (d) Give a definition of the Jacobson radical $J(R)$ of a ring R.

   (e) Show that $R/J(R)$ is semi-simple, quoting explicitly any needed results which are not proved.

   (f) Let K be a field and G a finite group.

   (i) Prove that if K has characteristic zero, then $K(G)$, the group algebra, is semi-simple.

   (ii) If char (K) is a prime p, for which G is $K(G)$ semi-simple? (Justify).

3. (a) Let p be a prime. Show that $x^2 + 1 \in Z_p[x]$ is irreducible iff $p \equiv 3 \bmod 4$.

   (b) Quoting without proof any needed results on fields and polynomials in general, prove that two finite fields of the same order are isomorphic.

   (c) Calculate the Galois group over $\mathbb{Q}$ of $(x^4 - 4)(x^2 + 8)$, explaining all steps carefully.

   (d) Prove that if K is an extension field of a field F, then there exists a unique intermediate field which is algebraic over F and is maximal among such intermediate fields.

(e) Define the term **algebraically closed**. Prove that a field F is algebraically closed iff, for every extension K of F, the intermediate field described in (d) is F itself.

(f) Define the algebraic closure of a field and sketch a proof that every field has an algebraic closure.

4. (a) Let V be a finite-dimensional complex inner product space.

   (i) State the spectral theorem concerning certain operators on V.

   (ii) Let $S : V \to V$ be an invertible linear operator on V such that $(S^*S)^2$ commutes with every operator on V. Show that S^*S must commute with every operator on V.

   (b) Find square matrices B and C which are not similar but which have the same characteristic and minimal polynomials.

   (c) Find a Jordan and a rational canonical form for

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

   (d) (i) Define the tensor product $V \otimes_K W$ of two vector spaces V and W over a field K.

   (ii) Prove that every element of $V \otimes_K W$ is of the form $v \otimes w$ iff at least one of $\dim V$ or $\dim W$ is less than 2.

   (e) (i) Define the exterior powers $\wedge^k V$ of a vector space V.

   (ii) Indicate how these spaces are related to determinants.

   (iii) Given a basis for V, describe a basis for $\wedge^k V$ and deduce the relationship between dim V and dim $(\wedge^k V)$.