Department of Pure Mathematics
Algebra Comprehensive Examination

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Time: 3 hours

Instructions: Answer 6 questions with at least one from each part.

1. Linear and Multilinear Algebra

1. (a) Find the Jordan canonical form of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & -2 & -3 \end{pmatrix} .$$

   (b) For what values of n is A^n similar to A? Give reasons.

   (c) Let B be a self-adjoint matrix. The Rayleigh quotient for the nonzero vector x is the scalar

$$R(x) = \frac{< Bx, x >}{< x, x >} .$$

   Prove that $\max_{x \neq 0} R(x)$ is the largest eigenvalue of B.

2. Prove that every complex n by n matrix is similar to its transpose.

3. Let V be an n-dimensional vector space over a field F, char $(F) \neq 2$ and let $f : V \times V \to F$ be a non-degenerate skew-symmetric bilinear form.

   (a) Prove that n is even, say $n = 2m$.

   (b) Prove that V has a basis $v_1, \ldots, v_{2m}$ such that $f(v_{2i-1}, v_{2i}) = -f(v_{2i}, v_{2i-1}) = 1$, $i = 1, \ldots, m$ while all other $f(v_i, v_j) = 0$.

2. Groups

4. Let us say that a group G has property (P) if it has a normal subgroup $N \simeq C_4 \times C_2$ and such that $G/N \simeq C_2 \times C_2$. ($C_k$ denotes a cyclic group of order k.)

   (a) Find all abelian groups G with property (P).

   (b) Give an example of a non-abelian group with property (P).

5. Let G be a finite p-group, p a prime, and let Φ be its Frattini subgroup, i.e., the intersection of all maximal subgroups of G.

   (a) Prove that $\overline{G} := G/\Phi$ is an elementary abelian p-group, that is, a direct product of cyclic groups of order p.

   (b) Prove that $x_1, \ldots, x_m \in G$ generate G if and only if their images $\bar{x}_1, \ldots, \bar{x}_m$ in $\overline{G}$ generate $\overline{G}$.

6. Let F_n be a free group of rank n.

   (a) If N is the subgroup of F_n generated by all squares x^2 with $x \in F_n$, prove that F_n/N is an elementary abelian 2-group and find its order.

   (b) If $F_m \simeq F_n$ prove that $m = n$.

3. Rings

(All rings have 1, which is inherited by subrings, and preserved by homomorphisms.)

7. Let F be a free right module of rank 1 over a ring R.

   (a) Prove that the ring $\mathrm{End}_R(F)$ is isomorphic to R.

   (b) If F^n is the direct sum of n copies of F, prove that $\mathrm{End}_R(F^n)$ is isomorphic to the ring $M_n(R)$ of n by n matrices over R.

   (c) If R is a division ring, prove that $M_n(R)$ is a simple artinian ring.

8. Let $I + J = A$ where A is a commutative ring and I, J ideals of A. Prove that

   (a) $IJ = I \cap J$,

   (b) $A/IJ \simeq (A/I) \times (A/J)$ (as rings) ,

   (c) $I^n + J^n = A$ for $n \geq 1$.

9. Prove that if a commutative ring A is noetherian then so is the polynomial ring $A[X]$.

4. Fields

10. (a) Prove that the regular pentagon can be constructed by ruler and compass.

    (b) Which of the following are generators of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$?
    (i) $\sqrt{2} + \sqrt{3}$, (ii) $\sqrt{2/3}$

    (c) Find the irreducible polynomials of $\sqrt{2} + \sqrt{3}$ and $\sqrt{2/3}$ over $\mathbb{Q}(\sqrt{2})$.

11. Let F be a finite field of order $q = p^n$, p a prime.

    (a) If $f, g \in F[x]$ are irreducible polynomials of the same degree and $E = F(\alpha)$ where $f(\alpha) = 0$, prove that g has a root in E.

    (b) Compute the number of monic quadratic irreducible polynomials in $F[x]$.

12. Let $k_1, \ldots, k_n > 1$ be square-free, pairwise coprime integers and $F_n = \mathbb{Q}(\sqrt{k_1}, \ldots, \sqrt{k_n})$.

    (a) Use induction on n to prove that $[F_n : \mathbb{Q}] = 2^n$.

    (b) Prove that the Galois group of $F_n/\mathbb{Q}$ is elementary abelian.