DEPARTMENT OF PURE MATHEMATICS

ALGEBRA COMPREHENSIVE EXAM

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Time: 3 hours

Instructions: Attempt 3 questions from F1, F2, G1, G2; and attempt 3 questions from R1, R2, L1, L2.

FIELDS (mostly)

F1. (a) Is it possible to construct, with straightedge and compass (starting as usual with two points distance 1 "unit" apart), an isosceles triangle with perimeter 8 units and area 2 square units?

(b) Determine (up to isomorphism) the Galois group of $x^3 - 3x - 3$ over $\mathbf{Q}$.

(c) Determine the number of subfields of the splitting field of $x^3 - 3x - 3$ over $\mathbf{Q}$.

F2. (a) Let p be a prime. For each positive integer k, define $f_k(x) = x^{p^k} - x$. Prove :

$$f_k \mid f_n \text{ in } \mathbf{Z}_p[x] \iff k \mid n \ .$$

(b) Suppose given a tower of three fields, $F \subset L \subset K$, in characteristic zero, such that

(i) $[\, L : F \,]$ is a power of 2, and

(ii) $[\, K : L \,]$ is an odd integer greater than 1.

Prove that there exists an extension E of F such that $[\, E : F \,]$ is an odd integer greater than 1.

GROUPS (mostly)

G1. Prove that no group of order 4125 is simple, using the Sylow theorems.

G2. (a) Suppose that H is a subgroup of a finite group G. Assume that, for some prime p, the order of H is p^m, and that of G is divisible by p^{m+1}. Prove that H is a proper subgroup of the normalizer $N_G(H)$.

(b) Determine all abelian groups of order 15 or less which are the underlying abelian group of a module over the following subring S of $\mathbf{Q}$:

$$S = \{\ a/3^k \mid a \in \mathbf{Z}\ ,\ k \in \mathbf{Z}\ ,\ k \geq 0\ \}\ .$$

RINGS (mostly)

R1. (a) Define a subring T of $\mathbf{Q}$ as follows:

$$T = \{\ a/b \mid a,\ b \in \mathbf{Z}\ ;\quad \text{and } b \text{ is odd}\ \}\ .$$

(i) Find $J(T)$, the Jacobson radical.

(ii) Describe $T/J(T)$ up to isomorphism.

(b) Give an example of a pair of integral domains A and B such that :

(i) as abelian groups, $(A,+) \cong \mathbf{Z} \oplus \mathbf{Z} \cong (B,+)$;

(ii) the group of invertibles in A is finite ; and

(iii) the group of invertibles in B is infinite.

R2. (a) Suppose that R is a commutative ring with identity such that every submodule of a free R-module is free. Prove that R is a principal ideal domain.

(b) Let G be a finite group, and F a field of characteristic zero. Prove Maschke's theorem : if M is a module over the group algebra of G over F, and if N is a submodule, then N is a direct summand in M. (Equivalently, show that the group algebra is semisimple.)

LINEAR ALGEBRA (mostly)

L1. Let V be a real vector space [of finite dimension at least 2, for parts (b) and (c)].

(a) Show that there is a unique linear operator T on $V \otimes V \otimes V$ which satisfies

$$T(u \otimes v \otimes w) \;=\; w \otimes u \otimes v$$

for all $u,\ v,\ w$ in V.

(b) Compute the dimensions of all the eigenspaces of T in terms of the dimension of V.

(c) Let S be the operator on $V \otimes V \otimes V$ which satisfies

$$S(u \otimes v \otimes w) \;=\; v \otimes u \otimes w\ ,$$

Determine the dimension of

$$(+1)\text{-eigenspace of } T \;\cap\; (-1)\text{-eigenspace of } S\ .$$

L2. (a) Denote the real $n \times n$ identity and zero matrices by I_n and O_n respectively. For non-negative integers a, b and c, let $J(a,b,c)$ be the block diagonal matrix

$$\begin{pmatrix} I_a & & \\ & -I_b & \\ & & O_c \end{pmatrix}.$$

Suppose that $J(a',b',c') \;=\; P^{\mathrm{tr}} J(a,b,c) P$ for some invertible P, where P^{tr} denotes its transpose. Prove that $a = a'$, $b = b'$ and $c = c'$.

(b) Deduce that a real symmetric matrix is congruent to at most one such matrix $J(a,b,c)$.