PURE MATHEMATICS

ALGEBRA

COMPREHENSIVE EXAMINATION

TIME: 3 HOURS Fall 1994

Try all the questions. More weight is given to completed answers than to many part answers.

LINEAR ALGEBRA

1. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V and let $v \in V$ be a vector such that $T^k(v) = 0$, but $T^{k-1}(v) \neq 0$. Find a subspace W of V of dimension k, invariant under T, so that T restricted to W is nilpotent of index k.

2. Let T be a linear operator on a finite dimensional complex inner product space V and let T^* be the adjoint of T.

   (a) Define a normal linear operator.

   (b) If T is normal and $v \in V$, prove that $T^*(v) = 0$ whenever $T(v) = 0$.

   (c) Prove that T is normal if and only if T commutes with TT^*.

GROUP THEORY

3. A group G is called *metacyclic* if it has a normal subgroup N such that N and G/N are cyclic.

   (a) Prove that every subgroup of a metacyclic group is metacyclic.

   (b) Give an example of a non-Abelian metacyclic group G together with the cyclic groups N and G/N.

4. Let A be the Abelian group formed by taking the direct sum of cyclic groups of orders $d_1, \ldots, d_k$, where $d_k > 1$ and d_{j+1} divides d_j for $1 \leq j < k$. Prove that no set of $k - 1$ elements of A can generate A.

RING THEORY

5. (a) Define: Euclidean domain.
   (b) Define: principal ideal domain.
   (c) Define: unique factorization domain.
   (d) Prove: A Euclidean domain is a principal ideal domain.
   (e) Prove: A principal ideal domain is a unique factorization domain.
   (f) Give an example of a unique factorization domain that is not a principal ideal domain, and prove that it is not a principal ideal domain.

6. (a) Show that $\mathbb{Z}\left[\frac{1+i\sqrt[3]{2}}{2}\right]$ is a Euclidean domain, using the square of the absolute value of an element.
   (b) Factor 93 into irreducibles in $\mathbb{Z}\left[\frac{1+i\sqrt[3]{2}}{2}\right]$.

FIELD THEORY

7. In this question we assume K is a finite extension of F. We define K/F to be *normal* if every polynomial $f(x) \in F[x]$ with a root in K actually splits in K. Aut(K/F) is the group of automorphisms of K/F. Fix(Aut(K/F)) is the set of elements of K which are left fixed by all $\alpha \in$ Aut(K/F).

   (a) i. Define: K is a separable extension of F.
   ii. If Fix(Aut(K/F)) = F show that for any $a \in K$

$$\prod_{\sigma \in \mathrm{Aut}(K/F)} (x - \sigma(a)) \in F[x].$$

   (b) In the following you may assume that for $a, b \in K$, if a and b are conjugate in K/F then there is an automorphism $\alpha \in$ Aut(K/F) such that $\alpha(a) = b$.
   i. Prove that Fix(Aut(K/F)) = F implies K/F is normal and separable.
   ii. Prove that if K/F is normal and separable then Fix(Aut(K/F)) = F.

8. (a) Find $[\mathbb{Q}(i + \sqrt[3]{2}) : \mathbb{Q}]$.
   (b) Find Aut($\mathbb{Q}(i + \sqrt[3]{2})/\mathbb{Q}$).
   (c) Determine the lattice of subfields of $\mathbb{Q}(i + \sqrt[3]{2})$.
   (d) Which of these subfields are normal extensions of $\mathbb{Q}$?
   (e) Which of these subfields are separable extensions of $\mathbb{Q}$?