UNIVERSITY OF WATERLOO

COMPREHENSIVE EXAMINATION IN ALGEBRA
MAY 1995

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Answer 6 of the following questions, including one question from each of the four parts indicated. Explain all your answers.

Part I - Linear algebra

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $det(T) = 1$ and $||Tx|| = ||x||$ for each x in $\mathbb{R}^3$.

   (a) Show that 1 is an eigenvalue of T.

   Let V be the eigenspace for the eigenvalue 1, and $V^\perp$ its orthogonal complement.

   (b) Show that $V^\perp$ is T-invariant and that V cannot be 2-dimensional.

   (c) Show that there is an orthonormal basis v_1, v_2, v_3 of $\mathbb{R}^3$ such that the matrix of T with respect to this basis is

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

   for some suitable angle θ.

   (d) Let S, T be linear operators on $\mathbb{R}^3$ such that each has a matrix representation of the above type with respect to suitable (but possibly different) orthonormal bases. Does ST have a matrix representation of the above type?

2. Let $M_n(K)$ be the ring of $n \times n$ matrices with coefficients in a field K of characteristic 0.

   (a) Show that the K-algebra $M_n(K)$ is simple.

   (b) Show that every K-algebra endomorphism $\psi : M_n(K) \to M_n(K)$ is a bijection. Is this true for ring endomorphisms of $M_n(K)$?

   (c) Find all K-linear maps $\psi : M_n(K) \to K$ such that $\psi(I) = 1$ and $\psi(AB) = \psi(BA)$ for all $A, B \in M_n(K)$, and I the identity matrix.

Part II - Ring Theory

3. A trigonometric polynomial is given by

$$a_0 \ + \ \sum_{k=1}^{n} (a_k \cos\, kt + b_k \,\sin\, kt)$$

where $a_i, b_i \in \mathbb{R}$. It has degree n if $a_n \neq 0$ or $b_n \neq 0$. It is known that for nonzero $p, q \in R$, the degree of pq is the sum of the degrees of p and q.

   (a) Explain briefly why the ring $R = \mathbb{R}[\cos \mathrm{t}, \sin \mathrm{t}]$ consists precisely of all trigonometric polynomials.

   (b) Use $\cos^2 t + \sin^2 t = 1$ to prove that R is not a unique factorization domain.

   (c) Prove that $R \cong \mathbb{R}[\mathrm{x}, \mathrm{y}]/(\mathrm{x}^2 + \mathrm{y}^2 - 1)$ as $\mathbb{R}$-algebras.

   (d) Prove that for nonzero $p, q \in R$, the degree of pq is the sum of the degrees of p and q.

4. (a) Let R be a left Artinian ring with identity element, and J its Jacobson radical. Prove that J is nilpotent.

   (b) Classify the 2-dimensional, associative, commutative algebras with identity element over the field $\mathbb{C}$ of complex numbers.

   (c) Repeat problem (b) in the 3-dimensional case.

Part III - Field Theory

5 (a) Prove that the order of a finite field is a power of a prime.

   (b) Construct a field of order 27, as a quotient of the polynomial ring $\mathbb{Z}_3[\mathrm{x}]$.

   (c) Show that the non-zero elements of a finite field form a cyclic group under multiplication.

6. Let $\zeta = e^{\pi i/7}$.

   (a) Show that $\mathbb{Q}(\zeta)$ is the splitting field of the polynomial $\mathrm{x}^{14} - 1$ over $\mathbb{Q}$.

   (b) What is the minimal polynomial of ζ over $\mathbb{Q}$ and the degree of the extension $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$?

   (c) Describe the Galois group of the extension $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$, its subgroups and the subfields of $\mathbb{Q}(\zeta)$ in Galois correspondence with these subgroups.

   (d) Explain whether or not a regular 14 sided polygon is constructible with ruler and compass.

Part IV - Group Theory

7. (a) Prove the theorem that a subgroup of a finitely generated abelian group is also finitely generated.

   (b)(i) Define the general semidirect product construction of two groups.

   (ii) Using a semidirect product (or otherwise) show that a subgroup of a finitely generated solvable group need not be a finitely generated group.

8. Let D_n denote the dihedral group of order $2n$.

   (a) Which D_n, if any, are non-trivial direct products?

   (b) Which D_n, if any, are nilpotent groups?

   (c) Compute the number of conjugacy classes of D_n, and give a representative for each class.

   (Justify your answers.)