Department of Pure Mathematics
Algebra Comprehensive Examination

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Answer at least six questions, including at least one question from each of the four sections.

I. Group Theory

1. (a) Give an example of a nontrivial group which has no maximal [proper] subgroups.

   (b) Consider the following property of a finite group: if H, K are distinct maximal subgroups, then $H \cap K = \{e\}$.

      i. Describe explicitly all finite *abelian* groups having the stated property.

      ii. Suppose that G is a finite group having the stated property, H is a maximal subgroup of G, and $H \ntriangleleft G$. Prove that

$$\left| \bigcup_{x \in G} xHx^{-1} \right| \geq \frac{1}{2}|G| + 1.$$

2. (a) Prove that every finite p-group is solvable.

   (b) Is there any analogue of (a) concerning groups G for which $|G|$ is divisible by exactly two primes? Explain.

   (c) Without using any theorem mentioned in your answer to (b), prove that if p is a prime larger than 3, then every group of order $4p^m$ $(m \geq 1)$ is solvable.

II. Ring Theory

3. (a) Give an example (with justification) of a Noetherian integral domain R and an ideal I of R which cannot be generated by fewer than 3 elements.

   (b) Prove that, in any Noetherian integral domain, every element is either 0, invertible, or a product of irreducible elements.

   (c) Give an example (with justification) of an integral domain R and an element $r \in R$ which is neither 0, invertible, nor a product of irreducible elements of R.

4. (a) Let R be the ring of all matrices $\begin{pmatrix} q & x \\ 0 & y \end{pmatrix}$ where q is a rational number and x, y are real numbers.

      i. Exhibit a composition series for R_R. treating R as a left module of itself

      ii. Show that $_RR$ is not Noetherian.

(b) Let R be a ring with unit such that

i. There exists a simple faithful left R-module;

ii. For every $r \in R$ there exists $n \geq 2$ such that $r^n = r$.

Prove that R is a division ring.

III. Field Theory

5. (a) Let F, K be fields with $F \leq K$ and $[K : F]$ finite. Suppose there are only finitely many subfields of K containing F. Prove that $K = F(c)$ for some $c \in K$.

(b) Give an example of fields $F \leq K$ such that $[K : F]$ is finite, K is a normal extension of F, but K is not a Galois extension of F.

(c) Suppose that F is an algebraically closed field of cardinality $2^{\aleph_0}$ and characteristic 0. Prove that F is isomorphic to the field $\mathbb{C}$ of complex numbers.

6. (a) It is known that the Galois group of $x^4 - x - 1$ over $\mathbb{Q}$ is S_4. Using this fact, or otherwise, give an example (with justification) of a field K satisfying $\mathbb{Q} \leq K \leq \mathbb{R}$, $[K : \mathbb{Q}] = 4$, and such that there does not exist a field F with $\mathbb{Q} < F < K$.

(b) Prove that if K is as in (a), then there is no normal extension E of $\mathbb{Q}$ satisfying $K \leq E$ and $[E : \mathbb{Q}] = 2^n$.

(c) If K is as in (a), and if $c \in K \setminus \mathbb{Q}$, does it follow that c cannot be constructed by straightedge and compass? Explain.

IV. Linear Algebra

7. (a) Find the smallest $n \in \mathbb{N}$ for which there exist seven matrices $A_1, A_2, \ldots, A_7$ in $\mathrm{Mat}_n(\mathbb{C})$ satisfying both (i) and (ii) below:

i. For each i, the eigenvalues of A_i are exactly the numbers 2, -2 and 3;

ii. No two of $A_1, \ldots, A_7$ are similar in $\mathrm{Mat}_n(\mathbb{C})$.

(b) Find the smallest $n \in \mathbb{N}$ for which there exist *symmetric* matrices $B_1, \ldots, B_7$ in $\mathrm{Mat}_n(\mathbb{R})$ satisfying both (i) and (ii) below:

i. For each i, the eigenvalues of B_i are exactly the numbers 2, -2 and 3;

ii. No two of $B_1, \ldots, B_7$ are orthogonally equivalent.

(c) Give an example of two linear operators $\mathbb{C}^n \to \mathbb{C}^n$ (for some n) which have the same minimal polynomials, same characteristic polynomials, but are not similar in the ring $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$.

(d) Let V, W be finite dimensional vector spaces over a field F. Prove that there exists a natural isomorphism $V^* \otimes_F W \to \mathrm{Hom}_F(V, W)$.

8. Let V and W be finitely generated $\mathbb{Z}$-modules. Let the invariant factors of V be $d_1, d_2, \ldots, d_k$, and let the invariant factors of W be $e_1, e_2, \ldots, e_l$ (possibly some are 0). Describe how to compute the invariant factors of $V \otimes_{\mathbb{Z}} W$.