Department of Pure Mathematics

Algebra Comprehensive Exam

May 28, 1996

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Time: 3 hours

Instructions: Do at least two questions from each of the four sections. However attempt to do as many of the 11 questions as possible.

1. Fields

1. Are $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ isomorphic? Give reasons.
2. (Splitting Field/Galois group)

   (a) Find the Galois group $\mathbf{G}$ of the splitting field $\mathbf{F}$ of $(x^2+3)(x^2+7) \in \mathbb{Q}[x]$.

   (b) Explicitly exhibit the galois correspondence between the subfields of $\mathbf{F}$, given as iterated radical extensions of $\mathbb{Q}$, and the subgroups of $\mathbf{G}$.
3. (Finite Fields).

   (a) Describe the factorization of $x^{p^n} - x$ over $\mathbb{Z}_p[x]$ into irreducible polynomials.

   (b) Use the above to calculate the number of irreducible monic polynomials in $\mathbb{Z}_3[x]$ of degree 4.

2. Groups

4. (a) Define what it means for $\mathbf{G}$ to be a free group.

   (b) Prove that $(\mathbb{Q}, +)$ is not free abelian.
5. Prove that a finite group $\mathbf{G}$ of order n is nilpotent if and only if it has a normal subgroup of order m for each divisor m of n.
6. (Sylow Theorems) Using the Sylow theorems, prove that no group of order 80 is simple.

3. Rings

All rings have 1. Subrings inherit the 1.

7. (Prime Ideals)

   (a) Define a prime ideal (for a commutative ring).

   (b) Show that if $\mathbf{R}$ is a commutative domain, S is a multiplicatively closed subset of $\mathbf{R}$ with $1 \in S$, but $0 \notin S$, and I is an ideal of $\mathbf{R}$ maximal with respect to disjointness from S, then I is a prime ideal.

   (c) Use the above to prove that an integral domain is a unique factorization domain if and only if every nonzero prime ideal contains a nonzero principal prime ideal.
8. Let $\mathbf{R}$ be a ring with $J(\mathbf{R}) = (\mathbf{0})$ (Jacobson radical). If $M_n(\mathbf{R})$ is the ring of $n \times n$ matrices over $\mathbf{R}$, prove that $J(M_n(\mathbf{R})) = (\mathbf{0})$.

9. Let $M_n(K)$ be the K-algebra of $n \times n$ matrices over a field K. Prove that $M_m(K)$ is isomorphic to a K-subalgebra of $M_n(K)$ if and only if m divides n. [Hint: Consider the irreducible modules.]

4. Linear Algebra

10. You are given a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in $\mathbf{F}[\mathrm{x}]$, where $\mathbf{F}$ is a field.
    (a) Define $\mathbf{C(f)}$, the companion matrix of f.
    (b) Prove that f is the minimal polynomial of $\mathbf{C(f)}$.
    (c) Prove that f is the characteristic polynomial of $\mathbf{C(f)}$.
    (d) Find a matrix with minimal polynomial $(x^2+2x+1)(x^2+2)$ and characteristic polynomial $(x^2+2x+1)(x^2+2)^2$.
11. (Inner Products)
    (a) Let $Q(x,y,z) = x^2 + 2y^2 + z^2 + 4xy - 6xz + 4yz$. Is there an orthonormal basis $\mathcal{B}$ with respect to which the quadratic form $Q(x,y,z)$ has the form $Q'(u,v,w) = au^2 + bv^2 + cw^2$? If so, find such a $\mathcal{B}$ and $Q'(u,v,w)$.
    (b) Hadamard's Inequality says that for vectors $\mathbf{u_1}, \ldots, \mathbf{u_n}$ in $\mathbb{R}^n$ one has

$$|D(\mathbf{u_1}, \ldots, \mathbf{u_n})| \leq \|\mathbf{u_1}\| \cdots \|\mathbf{u_n}\|$$

where D is the determinant function, and $\|\mathbf{u}\|$ is the Euclidean norm of $\mathbf{u}$. Prove Hadamard's Inequality.