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PURE MATHEMATICS

ALGEBRA

COMPREHENSIVE EXAMINATION

TIME: 3 HOURS

Spring 1992

Answer **six** questions and include one from each of the topics: Group Theory, Linear Algebra, Ring Theory, and Field Theory. Be sure to justify each of your answers.

Linear Algebra

1. Let T be a normal operator on a finite-dimensional complex inner product space V.

   (a) Show that $\|Tu\| = \|T^*u\|$ for all $u \in V$.

   (b) If $T^2u = 0$ show that $Tu = 0$.

   (c) Prove that the eigenvalues of T are all equal if and only if $T = cI$ for some number c, where I is the identity transformation.

2. (a) Let A be a real symmetric $n \times n$ matrix with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

   For any non-zero vector $x \in \mathbb{R}^n$, show that

$$\lambda_1 \leq \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \leq \lambda_n.$$

   (b) Find a matrix A with 1 as the largest modulus of its eigenvalues and such that

$$\frac{\langle x, Ax \rangle}{\langle x, x \rangle} > 1$$

   for some non-zero vector x.

   (c) If $\{w_1, \ldots, w_m\}$ is an orthonormal set in a finite-dimensional inner product space V such that

$$\sum_{i=1}^{m} |\langle w_i, v \rangle|^2 = \|v\|^2 \quad \text{for every } v \in V$$

   prove that $\{w_1, \ldots, w_m\}$ is a basis of V.

Group Theory

3. (a) If a finite group G acts on a set X and $x \in X$, prove that the number of elements in the orbit of x is the index of the stabilizer of x in G.

   (b) The general linear group, $GL_2(\mathbb{Z}_p)$, of invertible 2×2 matrices over the finite field $\mathbb{Z}_p$, acts on all the 2×2 matrices over $\mathbb{Z}_p$ by left multiplication of matrices. Determine the order of the stabilizer subgroup of each 2×2 matrix.

4. (a) State the first Sylow Theorem.

   (b) If G is a group of order pq, where p and q are distinct primes, prove that either G is cyclic or G is generated by two elements a and b satisfying

$$b^p = 1; \quad a^q = 1; \quad a^{-1}ba = b^r.$$

Ring Theory (NOTE: All rings are to be considered as rings with identity.)

5. (a) Prove that principal ideal domains are unique factorization domains.

   (b) Let R be an integral domain. When is the polynomial ring

$$R[x_1, x_2, \ldots, x_n], \quad (n \geq 1)$$

   a principal ideal domain? Justify your answer. When is $R[x_1, x_2, \ldots, x_n]$ a unique factorization domain?

   (c) Show that $\mathbb{Q}[\sqrt{-1}]$ is a Euclidean domain.

6. (a) Let R be a finite commutative ring and let $J(R)$ be the Jacobson radical of R. Describe $R/J(R)$. Show that there is an integer $n \geq 2$ so that every element of this quotient ring satisfies $x^n = x$.

   (b) If R is a ring whose Jacobson radical is zero, prove that the Jacobson radical of a matrix ring over R is zero; i.e. $J(R) = 0$ implies $J(M_n(R)) = 0$.

Field Theory

7. (a) Outline the key ideas in the proof that for every prime p and positive integer n there is, up to isomorphism, exactly one field of size p^n.

   (b) Find the monic polynomial of smallest degree in $\mathbb{Z}_2[x]$ which vanishes on both $\mathbf{GF}(4)$ and $\mathbf{GF}(8)$.

8. (a) What conditions does the field extension $\mathbb{Q}(\alpha)$ have to satisfy for $\alpha \in \mathbb{R}$ to be constructible by straight edge and compass? Determine whether the number $\alpha = \cos \frac{\pi}{9}$ is constructible.

   (b) Prove that the automorphism group of the algebraic closure of $\mathbb{Q}$ has the cardinality of the continuum.

   (c) Describe the automorphism group of the field of real numbers.