Out of 84.

Department of Pure Mathematics
Algebra Comprehensive Examination

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Attempt each of problems 1 through 6.

1. Let A be an $n \times n$ matrix over $\mathbb{C}$ such that $I_n, A, A^2, \dots, A^{n-1}$ are linearly independent. Show that there exists $x \in \mathbb{C}^n$ such that the
8 vectors $x, Ax, A^2x, \dots, A^{n-1}x$ are linearly independent.

2. Let R be a principal ideal domain and A an $n \times n$ matrix over R. Two $n \times n$ matrices A, B over R are *equivalent* over R if there exist invertible matrices P, Q over R such that $B = PAQ$.

   (i) Define the following terms for A :

   1 a) determinantal divisors

   1 b) invariant factors

   1 c) Smith normal form.

   (ii) Find the Smith normal form of $A = I_n - B$ if

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$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$$

   where the a_i are elements of R and I_n is the $n \times n$ identity matrix.

   (iii) Prove that two $n \times n$ matrices over R are equivalent if and only
   5 if they have the same determinantal divisors.

3. Let H be a subgroup of a cyclic group G. Is it true that every automorphism of H extends to an automorphism of G? Prove your claim.

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4. Let G be the group generated by the linear operator T in $\mathbb{R}^5$ defined by $T(e_k) = e_{k+1}, 1 \leq k \leq 4$, and $T(e_5) = e_1$, where $e_1, \ldots, e_5$ is the standard basis of $\mathbb{R}^5$. Decompose $\mathbb{R}^5$ into irreducible invariant subspaces of G. 10

5. For any ring R, let $GL_n(R)$ be the group of invertible $n \times n$ matrices over R. In this problem we set $R = \mathbb{Z}/p^k\mathbb{Z}$ where p is a prime and k is a positive integer.

6 (a) Compute the order of $GL_n(R)$.

6 (b) For which values of n, p, k is $GL_n(R)$ solvable?

Hint: Consider first the case $k = 1$. In the general case use the canonical homomorphism $\mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

6. $\mathbb{Q}$ denotes the rational number field .

3 a) Prove that any extension of $\mathbb{Q}$ of degree 2 has the form $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Z}$.

3 b) If E is an extension of a field F such that every element of E is algebraic over F, is E a finite extension of F? Prove your answer.

4 c) Let α be a transcendental element over $\mathbb{Q}$. Determine whether the set

$$\{\alpha^k | k \in \mathbb{Z}\}$$

is a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

Attempt one of problems 7 and 8.

7. Let A and B be associative algebras with identity over a field F. 12

5 (a) Show that there is ~~a unique~~ an F-algebra structure on $A \otimes_F B$ such that

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

for all $a, a' \in A$ and $b, b' \in B$.

7 (b) If $\mathbb{H}$ is the algebra of real quaternions show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic (as an $\mathbb{R}$-algebra) to $M_2(\mathbb{C})$, the algebra of 2×2 complex matrices.

12 8. Let R be a non-trivial ring with 1, and let J be the intersection of all maximal left ideals of R. Let M be a finitely generated left R-module.

4 (a) If M is simple show that $JM = 0$.

4 (b) ~~If $M \neq 0$ show that M has a maximal~~ submodule. Show that every proper submodule of M is contained in a maximal submodule of M.

4 (c) If $JM = M$ show that $M = 0$.

Attempt one of problems 9 and 10.

12 9. Let F be ~~a purely transcendental extension of~~ an algebraically closed field E. If σ is an automorphism of F show that $\sigma(E) = E$.

12 10. Determine whether $\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{5}, i)$ is a normal extension of $\mathbb{Q}(i)$. (Q denotes the rationals and i the imaginary unit.)

Let F be an alg. clsd field: If σ is an automorphism of $F(t)$, where t is transcendental over F, show that $\sigma(F) = F$.