PURE MATHEMATICS ALGEBRA
COMPREHENSIVE EXAMINATION

Spring 1993

Prepared by: J. Lawrence and F. Tang

Do 6 questions including at least one question from each of the four sections.

Notation	$\mathbb{Q}$	- set of rational numbers
	$\mathbb{R}$	- set of real numbers
	$\mathbb{C}$	- set of complex numbers
	$\mathbb{Z}_n$	- integers modulo n.

Section I - Rings

1. (a) Define: Noetherian ring.

   (b) State the Hilbert Basis Theorem.

   (c) State and prove Gauss's Lemma (over the integers).

   (d) If R is a unique factorization domain, describe all the units and primes of $R[t]$ (the polynomial ring) in terms of the units and primes of R.

2. The Jacobson radical of a ring is the intersection of the maximal left ideals of the ring.

   (a) Prove that the Jacobson radical of a commutative algebra which is finite-dimensional over a field is the set of nilpotent elements.

   (b) Let $F = \mathbb{Z}_2(\mathrm{t})$ be the purely transcendental extension over $\mathbb{Z}_2$ and let L be the subfield generated by t^2. Find a basis over L for the Jacobson radical, $J(R)$, of $R = F \otimes_L F$. What is $R/J(R)$?

   (c) State the Wedderburn-Artin Theorem.

   (d) What is the number of isomorphism classes of semisimple $\mathbf{C}$-algebras of dimension 9? Justify your answer.

Section II - Fields

3. (a) Define: "separable (finite) extension."

   (b) For which of the following fields is every finite extension separable?

   i) $\mathbb{Z}_5$,

   ii) $\mathbb{Z}_5(\mathrm{x})$ (rational functions over $\mathbb{Z}_5$),

   iii) $\mathbf{Q}$

   iv) $\mathbf{Q}(\mathrm{x})$. (rational functions over $\mathbf{Q}$)

   Briefly justify your answer, stating the basic theorems used.

(c) Let $F = \mathbb{Z}_p(t)$ be the transcendental extension of $\mathbb{Z}_p$ by t (where p is a prime). Let $f(x) = x^p - t$ and let K be the splitting field of $f(x)$ over F. Determine with reasons whether K is a separable extension of F.

4. (a) Define: "normal extension of the field F."

   (b) If K is a normal extension of F and E is a normal extension of K, is E necessarily a normal extension of F? Justify your answer.

   (c) Let $f(x) \in \mathbb{Q}[x]$. If $f(x)$ is irreducible of degree, $p \geq 5$, p a prime, in $\mathbb{Q}[x]$ and contains exactly two non-real roots, show that $f(x)$ is **not** solvable by radicals over $\mathbb{Q}$.

   (d) Describe the Galois group of $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$.

Section III - Groups

5. (a) Let G be an abelian group of order p^n, where p is a prime. If for all $x \in G, x^p = 1$, find the order of the automorphism group of G.

   (b) Define: "nilpotent group of (nilpotence) class c."

   (c) Let G be a nilpotent group of class c and let G' be its commutator subgroup. Show that $H = \langle G', x \rangle, x \in G$, is nilpotent of class at most $c - 1$.

   (d) Let p, q and r be distinct primes. If the order of G is pqr, show that G is solvable.

6. Let G be a finite group whose only characteristic subgroups are $\{1\}$ and G (G is characteristically simple). Show that G is isomorphic to a direct product of simple groups.

Section IV - Linear Algebra

7. a) Let $V = U_1 \oplus U_2 \oplus \ldots \oplus U_n$ be a vector space and let $E_i : V \to U_i$ be the projection map. If T is a linear operator on V, prove that U_i is T-invariant, for all i, if and only if T commutes with each E_i.

b) Let V be a finite dimensional complex inner product space and suppose that the operator T is one-to-one (injective). Let $S = TT^*$, where T^* is the adjoint of T. Show that S has real eigenvalues.

8. a) Let A be an $n \times n$ matrix over $\mathbb{C}$. Describe the Jordan canonical form of A.

b) Use the Jordan canonical form to show that every complex matrix is similar to its transpose.