UNIVERSITY OF WATERLOO

COMPREHENSIVE EXAMINATION IN ALGEBRA
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Do 6 questions with at least one from each part.

Part I - Group Theory

1. (a) List all the non-isomorphic abelian groups of order 200.

   (b) Prove that groups of order p^2 are abelian where p is a prime.

   (c) Prove that groups of order p^2q are solvable, where p, q are primes, not necessarily distinct.

2. (a) Define nilpotent groups.

   (b) Let G be a nilpotent group and H, a proper subgroup of G. Let $N_G(H)$ be the normalizer of H in G. Show that $H \subsetneq N_G(H)$.

   (c) Let G be a finite nilpotent group. Show that G is the direct product of its Sylow subgroups. (Hint: Use the fact that if P is a Sylow subgroup of G, $N_G(N_G(P)) = N_G(P)$.)

Part II - Linear Algebra and Matrix Theory

3. (a) Prove that any orthogonally diagonizable real $n \times n$ matrix is symmetric.

   (b) Let $A = (a_{ij})$ be any $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove that for each $i = 1, \ldots, n \quad \lambda_1 \geq |a_{ii}| \geq \lambda_n$.

4. (a) Find the Jordan Canonical form of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 3 \end{bmatrix} .$$

   (b) Is A similar to its transpose?

   (c) Show that if V is finite dimensional over $\mathbb{Q}$ then there exists a linear map $A : V \to V$ such that

$$x, A(x), \ldots, A^{m-1}(x) \qquad\qquad 1 < m = dim\ V$$

   are linearly independent for every nonzero $x \in V$.

(d) Is this still true over $\mathbb{C}$?

Part III - Rings and Modules

5. (a) Show that the ring of $n \times n$ matrices over any field is a simple ring.

   (b) A ring R of endomorphisms of a vector space V over a division ring D is said to be **dense** if for every linearly independent set $u_1, \ldots, u_n$ of vectors in V and any arbitrary set of vectors $v_1, \ldots, v_n$ in V there exists an endomorphism $\phi \in R$ such that $\phi(u_i) = v_i;\ i = 1, \ldots, n$. Prove that such an R is left Artinian if and only if $dim_D\ V$ is finite.

6. (a) Let R be a commutative ring with unity. Prove that any two bases of a finitely generated R-module have the same cardinality.

   (b) Let M be an R module over a commutative ring with unity. Let $f_1, \ldots, f_m \in \text{Hom}\ (M, R)$ and let

$$N = \bigcap_{i=1}^{m} \ker f_i \quad .$$

   Prove that M/N is finitely generated.

Part IV - Field Theory

7. Let F be a field and let K be an extension of F.

   (a) Explain what is meant by K is an algebraic extension of F.

   (b) Show that if $[K : F]$ is finite then K is an algebraic extension of F.

   (c) Show that the set of all algebraic elements of K over F is a field.

8. (a) Let F be a field and char $F \neq p$. Show that if $a \in F$ then $f(x) = x^p - a$ is either irreducible in $F[x]$ or $f(x)$ has a root in F.

   (b) Find the Galois group of $f(x) = x^4 - 2$ over $\mathbb{Q}$. If K is the splitting field of $f(x)$ over $\mathbb{Q}$ find all subfields of degree 4 in K which are normal over $\mathbb{Q}$.