The Comprehensive Examination, Part A - Analysis - for Graduate Students in the Department of Pure Mathematics

10 a.m.-1 p.m. October 31, 1978

Real Analysis

1. Suppose $f: [0,1]\times[0,1] \to R$ is continuous and $\epsilon > 0$. Show that there exists continuous functions $f_1,\dots,f_n,g_1,\dots,g_n$ from $[0,1]$ to R such that $\left|f(x,y)-\sum_{1=1}^{n} f_i(x)g_i(y)\right| < \epsilon$ for all $0 \le x,y \le 1$.

2. Use the Arzela-Ascoli theorem to prove that if $f_n: [0,1] \to R$ is a sequence of differentiable functions satisfying $|f_n(x)|+|f_n'(x)| \le 1 \quad \forall x$, then it has a uniformly convergent subsequence. Is the limit function necessarily differentiable?

Complex Analysis

3. Let f be an entire function, and let $u(x,y)$ and $v(x,y)$ be the real and imaginary parts of $f(x+iy)$, $x,y \in R$. Show that $w(x,y) = (u(x,y))^2 - (v(x,y))^2$ is harmonic in R^2.

4. a) Find the residue of $f(z) = \frac{1}{z^2+1}$ at all singularities.

   b) Use the Residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

5. Suppose f is nonconstant analytic in a neighbourhood of $\bar{D} = \{z: |z| \le 1\}$ and that $f(z) \ne 0$ on $\bar{D}$. Show that there is a point $e^{i\theta}$ so that $|f(z)| > |f(e^{i\theta})|$ for all $|z| < 1$.

Topology

6. a) Let X be a compact topological space and let Y be Hausdorff. Suppose $f: X \to Y$ is continuous, 1-1, and onto. Prove that f is a homeomorphism.

   b) Give a simple example to show that Y must be Hausdorff for this theorem to be correct.

Functional Analysis

7. Let $X = \ell_1^{(n)}$, the space $\mathbb{C}^n$ with norm of $\bar{a} = (a_1, \ldots, a_n)$ given by $||\bar{a}||_1 = \sum_{i=1}^{n} |a_i|$, and let $Y = \ell_2^{(n)}$ with norm $||\bar{a}||_2 = (\sum_{1}^{n} |a_i|^2)^{1/2}$. Let $T: X \to Y$ be defined by $T\bar{a} = \bar{a}$.

   a) Use the open mapping theorem to show that $\exists \mu_n, M_n > 0$ such that $\mu_n ||\bar{a}||_2 \le ||\bar{a}||_1 \le M_n ||\bar{a}||_2$.

   b) What happens as $n \to \infty$? Where does the proof in (a) fail for $n = \infty$ (i.e. for ℓ_1 and ℓ_2).

8. If X is a Banach space, define the natural imbedding $i: X \to X^{**}$. Show that if $y \in X^{**}$, then $y \in i(X)$ if and only if y is continuous in the weak*topology.