THE COMPREHENSIVE EXAMINATION IN ANALYSIS FOR GRADUATE STUDENTS IN PURE MATHEMATICS

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Real Analysis

1. Show that the series $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\ldots$ converges and find its sum. State clearly any theorems used to justify your method.

2. Construct sequences $\{f_n\}$ and $\{g_n\}$ of continuous real functions which converge uniformly on R but such that $\{f_n g_n\}$ does not converge uniformly on R.

3. If $f:(0,1) \to R$ is differentiable and $\lim_{x\to 0+} f'(x) = L$ exists, prove that $f(0+) = \lim_{x\to 0+} f(x)$ exists and $\lim_{x\to 0+} \frac{f(x)-f(0+)}{x} = L$.

Complex Analysis

4. Evaluate $\int_0^\infty \frac{\cos x}{x^2+1}\,dx$.

5. Prove that if f is an entire function and $f(x+1) = f(x)$ for all real x, then $f(z+1) = f(z)$ for all complex z.

6. Let $f(z) = z^n + a_{n-1}z^{n-1} +\ldots+ a_0$ and $g(z) = z^m + b_{m-1}z^{m-1} +\ldots+ b_0$ be two polynomials. Suppose that $\{z: |f(z)| > 1\}\subseteq\{z: |g(z)| > 1\}$. Prove that $f(z)^m = g(z)^n$. (Hint: Consider $h(z) = f^m(z)/g^n(z)$ in a neighbourhood of ∞).

Measure Theory

7. Prove that if $f\colon R \to R$ is differentiable then its derivative, f', is Lebesgue measurable.

8. (a) Given two functions $f, g \in L^1(R)$, show that the convolution $f*g$, defined by

$$f*g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)g(x-t)\,dt, \quad x \in R,$$

is a well defined member of $L^1(R)$ by using Fubini's theorem. (You may assume that the mapping $(x,t) \to f(t)g(x-t)$ is measurable).

(b) If $f \in L^1(R)$, the Fourier transform if f, $\hat{f}$, is defined by $\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ixs}dx$, $s \in R$. Prove that if $f, g \in L^1(R)$ then $\widehat{(f*g)}(s) = \hat{f}(s)\hat{g}(s)$ for all $s \in R$.

Convexity

9. Let K be a compact, convex subset of R^n with non-void interior and let $f\colon K \to R$ be continuous and convex.

(a) Prove that f attains its maximum at an extreme point of K.

(b) Prove that if f attains its maximum at an interior point of K then f is constant.