The Comprehensive Examination in Analysis - for Graduate students in Pure Mathematics and Combinatorics and Optimization

TIME: Monday, May 7, 1979

Do all the questions in each section that has been selected. Every student will do Real Analysis, Complex Analysis, and two other topics.

Real Analysis

1. If f is a differentiable function on R and $f'(0) < d < f'(1)$, show that there exists a real number c with $0 < c < 1$ and $f'(c) = d$.

   Hint: Consider $d = 0$ first.

2. Suppose a_k are complex numbers with $|a_k| \leq M$ for all $k = 1,2,\ldots$. Let

$$f(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin kx.$$

   Prove that the series converges for $t > 0$ to a C^2 function on $\{(x,t): x \in R, t > 0\}$ and satisfies the differential equation $f_t = f_{xx}$.

Complex Analysis

3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$.

4. Let $f(z)$ be an entire function satisfying $|f(z)| \leq c|z|^n$. Prove that f is a polynomial of degree less than or equal to n.

5. Let $p(z) = 1-2z+z^m$ where m is an integer greater than or equal to 3. Show that $p(z)$ has precisely one zero in the disc $D = \{z: |z|<1\}$.

Hint: Apply Rouché's theorem on the a cirlce of radius $1-\delta$ with $\delta > 0$ small.

Topology

6.a) State Urysohn's Lemma.

b) Prove it for metric spaces.

7. Let I be the product of countably many copies of the interval $[0,1]$ with the product topology. Let J be the same set with the metric topology given by $\rho((x_n),(y_n)) = \sum_{n=1}^{\infty} 2^{-n}|x_n-y_n|$. Show that I and J are homeomorphic.

Functional Analysis

8. Let T be a linear map defined on a Hilbert space H satisfying $(Tx,y) = (x,Ty)$ for all x,y in H. Prove that T is continuous.

9. Let $\{e_n\}$ be the standard basis for ℓ^1. Show that bounded linear operators on ℓ^1 are in a one to one correspondence with bounded sequences $\{f_n\}$ in ℓ^1 by $Te_n = f_n$. Calculate $\|T\|$ in terms of $\{f_n\}$. Show that T is the adjoint of an operator on c_0 if and only if f_n converges to 0 in the weak* topology.

Convex Sets

10. Let Δ be an n-dimensional simplex in L with vertices $x_1, x_2, \ldots, x_{n+1}$. (That is the flat of minimum dimension containing the x_i $(i = 1,\ldots,n+1)$ has dimension n and Δ is the convex hull of the x_i $(i = 1,\ldots,n+1)$) Show that Δ consists of all points x in L for which constants $\alpha_i \geq 0$ $(i = 1,\ldots,n+1)$ exist such that

$$x = \sum_{i=1}^{n+1} \alpha_i x_i, \qquad \sum_{i=1}^{n+1} \alpha_i = 1$$

holds.

11. Let Z be a closed convex set in R^m and let $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m)$ be a point of $\mathbb{R}^m$ not in Z. Show that there exists a hyperplane which passes through $\bar{w}$ and does not meet Z.