Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

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Time: 3 hours

1. a) Evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x^2+1}\,dx$.

   b) Find an analytic function which gives a one-to-one map from

   $$U = \{re^{i\theta} \mid 0 < \theta < \frac{\pi}{2}\ ,\ 0 < r < 1\}$$

   onto $H = \{z \mid Im\ z > 0\}$.

2. Recall that a topological space is normal if for each disjoint pair of closed sets A and B there are disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

   a) Show that a metric space is normal

   Let X be a compact Hausdorff space

   b) Show that X is normal.

   c) Given disjoint closed subsets $A, B \subseteq X$, construct a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

3. A set $B \subseteq \mathbb{R}$ is called a G_δ set if $B = \bigcap_{n=1}^{\infty} U_n$ where each U_n is open.

   a) Prove that $\mathbb{Q}$ is not a G_δ set.

   b) Show that Card $\{A \subseteq \mathbb{R} \mid \text{A is } G_\delta\} =$ Card $(\mathbb{R})$.

4. Let $C[0,1]$ denote the Banach space of continuous real valued functions on $[0,1]$ with the norm $||f||_\infty = \max\ \{|f(x)| \mid x \in [0,1]\}$. Define $T : C[0,1] \to C[0,1]$ by

   $$T(f)(x) = e^x + \frac{1}{2}\int_0^x \cos(f(t))\ dt$$

   a) Show that these exists a unique $f \in C[0,1]$ such that $T(f) = f$.

   b) Let $T^{n+1}(g) = T(T^n(g))$ for $n = 1, 2, \ldots$. Show that $\{T^n(g) \mid n \in \mathbb{N}\}$ is equicontinuous for each $g \in C[0,1]$.

5. a) Show that a holomorphic function on an open domain must map open sets onto open sets and derive the maximum modulus principle as a result.

Let $\mathbb{D} = \{z \mid |z| < 1\}$, $\mathbb{D}^* = \mathbb{D} \backslash \{0\}$.

b) Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$, then $|f'(0)| \leq 1$ with equality iff $f(z) = cz$ for some c which $|c| = 1$.

c) Show that every holomorphic function $f : \mathbb{D}^* \to \mathbb{D}$ is the restriction to $\mathbb{D}^*$ of a holomorphic function on $\mathbb{D}$.

6. a) Use the axiom of choice to construct a nonmeasurable set in $[0, 1]$.

b) Let $(\mathcal{X}, \mathcal{M}\mu)$ be a measure space. A map $T : \mathcal{X} \to \mathcal{X}$ is called measurable if $T^{-1}(E) \in \mathcal{M}$ for each $E \in \mathcal{M}$. T is said to be measure preserving if it is measurable and if $\mu(T^{-1}(E)) = \mu(E)$ for each $E \in \mathcal{M}$.

Prove Poincaré's Recurrence Theorem:

If $(\mathcal{X}, \mathcal{M}\mu)$ is a probability space (i.e., $\mu(\mathcal{X}) = 1$), T is a measure preserving transformation and $\mu(E) > 0$, then almost every $x \in E$ returns to E infinitely often under positive interations of T.

Hint: For $N \geq 0$, let $E_N = \cup_{n=N}^{\infty} T^{-n}(E)$, where $T^0(E) = E$. Show that $\mu(E_0) = \mu(E_n)$ for all n.

7. A sequence of continuous 2π-periodic functions $\{k_n\}$ is called a summability kernel if

(S1) $\displaystyle \frac{1}{2\pi} \int_0^{2\pi} k_n(t)\, dt = 1.$

(S2) There exists an $M < \infty$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |k_n(t)|\, dt \leq M \quad \text{for all } n \quad .$$

(S3) For every $0 < \delta < \pi$

$$\lim_{n \to \infty} \int_\delta^{2\pi - \delta} |k_n(t)|\, dt = 0 \quad .$$

a) Let f be a 2π-periodic function which is monotonic on $[0, 2\pi)$. Prove that

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} k_n(t) f(x - t)\, dt$$

exists for each $x \in [0, 2\pi)$.

b) Prove that $f(x) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} k_n(t) f(x-t)\, dt$ almost everywhere on $[0, 2\pi)$.

8. Let $\mathcal{M}_\lambda$ be the σ-algebra of Lebesgue measurable sets in $[0,1]$. Define an equivalence relation $\sim$ on $\mathcal{M}_\lambda$ by $A \sim B$ iff $\lambda(A\Delta B) = 0$ where $A\Delta B = (A \cap B^c) \cup (A^c \cap B)$ is the symmetric difference and λ is the Lebesgue measure. Define ρ on $(\mathcal{M}_\lambda/\sim) \times (\mathcal{M}_\lambda/\sim)$ by $\rho([A],[B]) = \lambda(A\Delta B)$.

   a) Show that $(\mathcal{M}_\lambda/\sim\ ,\ \rho)$ is a metric space.

   b) Let $\{A_n \mid n \in \mathbb{Z}\} \subset \mathcal{M}_\lambda$ be such that $A_n \subseteq A_{n+1}$. Show that $[A_n]$ converges as $n \to \infty$.

   c) Prove that $(\mathcal{M}_\lambda/\sim\ ,\ \rho)$ is complete.