Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

MAY 1995

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Time: 3 hours

Answer all the questions.

1. (a) Prove that if $p(z)$ is a non-constant polynomial with complex coefficients then p has at least one complex root.

   (b) Suppose f is analytic on the annulus $1 \leq |z| \leq 2$ and that $|f(z)| \leq 3$ on $|z| = 1$ and $|f(z)| \leq 12$ on $|z| = 2$. Prove that $|f(z)| \leq 3|z|^2$.

2. Discuss the convergence of the infinite products

$$\prod_{n=1}^{\infty} \left(1 + \frac{i}{n}\right) \quad \text{and} \quad \prod_{n=1}^{\infty} \left|1 + \frac{i}{n}\right| .$$

3. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx \;=\; \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$.

4. (a) Let A be a subset of a topological space X. Prove that A is open if and only if $\overline{A \cap \overline{B}} = \overline{A \cap B}$ for all $B \subseteq X$.

   (b) Let A and B be dense in a topological space X. Prove that if A is open then $A \cap B$ is dense.

5. Let $f_n \in C[0,1]$, for $n = 1, 2, 3, \ldots$, $f_n \stackrel{n\to\infty}{\longrightarrow} f$ pointwise and suppose $\{f_n\}_{n=1}^{\infty}$ is equicontinuous. Prove $f_n \to f$ uniformly.

6. (a) State the Baire category theorem.

   (b) Suppose $U = \text{span}\,\{x_n : n = 1, 2, \ldots\}$ is a closed subspace of a Banach space. Prove U is finite dimensional.

7. Show that $f(x,y) = \frac{xy}{(x^2+y^2)^2}$ defines a function which is not integrable on $[-1,1] \times [-1,1]$ with the usual Lebesgue product measure.

8. Let (Ω, μ) be a measure space, A a measurable subset of Ω, and let f be positive and μ-integrable. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A f d\mu < \epsilon$.

9. Let $P(X)$ be the power set of X and let $F : P(X) \to P(X)$ be monotone (i.e. $A \subseteq B \Rightarrow F(A) \subseteq F(B)$). Show there is a set $B \subseteq X$ with $F(B) = B$.

10. Let X be a compact, Hausdorff space and let Y be a T_1 space. Let $f : X \to Y$ be a continuous surjection. Show there is a compact set $X_0 \subseteq X$ such that $f(X_0) = Y$, but no proper, closed, subset of X_0 is mapped by f onto Y.