Department of Pure Mathematics
Analysis and Topology Comprehensive Examination

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Attempt each of the following 4 questions:

1. a) Define the notion of a quotient of a topological space X.

   b) Prove that if X, Y are topological spaces and $f : X \to Y$ is a continuous surjection which is either open or closed, then the topology on Y is the quotient topology.

   c) Let $X = [0, 1]$ with the usual topology. Let R_1 be the equivalence relation defined by identifying the points $\frac{1}{3}$ and $\frac{2}{3}$ and let R_2 be the equivalence relation defined by identifying the points $\frac{1}{2}$ and 1. Is X/R_1 homeomorphic to X/R_2? Explain.

2. a) State the Axiom of Choice.

   b) Construct a subset of $\mathbb{R}$ which is not Lebesgue measurable. Justify your construction.

3. a) State the Ascoli-Arzela Theorem.

   b) Use the Ascoli-Arzela Theorem to derive the Heine-Borel Theorem for $\mathbb{R}^n$ [n > 1].
   Hint: Think of $\mathbb{R}^n$ as the set of all functions from $\{1, 2, \ldots, n\}$ into $\mathbb{R}$.

4. a) i) What does it mean for an open subset of $\mathbb{C}$ to be simply connected.

   ii) State the Riemann Mapping Theorem.

   iii) Find an analytic map transforming the annulus $\{z | a < |z| < b, Re z > 0\}$ $(0 < a < b)$ onto an open rectangle.

b) Evaluate $\displaystyle\int_\gamma \frac{z^2-1}{(z^2+1)^2}\,dz$

where $r(t) = i + e^{it}$ for $0 \le t \le 2\pi$.

Attempt two of the following four problems:

5. a) Let $f \,\epsilon\, L^1[0,2\pi]$ be a function with Fourier coefficients

$$\hat{f}(n) = \begin{cases} \frac{1}{n^2 \log|n|} & \text{if } \ |n| \ge 2 \\ 0 & \text{otherwise} \end{cases} .$$

Show that $f \,\epsilon\, C[0,2\pi]$ but $f \not\epsilon\, C^2[0,2\pi]$.

b) Let $f, g \,\epsilon\, L^2[0,2\pi]$. Show that $f * g \,\epsilon\, C[0,2\pi]$ where

$$f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)dy.$$

6. a) Prove that a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(\frac{x+y}{2}) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \ \text{ for all } \ x, y \,\epsilon\, \mathbb{R}$$

if and only if for some $a, b \,\epsilon\, \mathbb{R}$, f(x) = ax + b.

b) Is the above true when continuity is replaced by Lebesgue measurability? Explain.

7. Show that the product space $[0,1]^{\mathbf{R}}$ is

a) not metrizable,

b) separable.

8. a) Let $A \subset \mathbb{C}$ be open. Let $f : A \to \mathbb{C}$ be continuous and such that $f = F'$ for some analytic function $F : A \to \mathbb{C}$. Let $\gamma : [0,1] \to A$ be a continuous, piecewise-$\mathbb{C}^1$ curve with $\gamma(0) = z_1,\ \gamma(1) = z_2$. Prove that

$$\int_\gamma f = F(z_2) - F(z_1)$$

b) Prove the Cauchy-Goursat Theorem for rectangles:
Let $A \subset \mathbb{C}$ be open. Let $f : A \to \mathbb{C}$ be analytic. Let $R = [a,b] \times [c,d]$ be a rectangle in A with perimeter γ. Then

$$\int_\gamma f = 0.$$

Attempt one of the following two problems:

9. Prove that if $A \subset [0,1]$ is such that

$$m^*(A) + m^*([0,1] \backslash A) = 1$$

where m^* is the outer Lebesgue measure, then A is Lebesgue measurable. (Recall: The Carathéodory definition for measurablility is that A is measurable if $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$ for any $E \subset [0,1]$).

10. Let $f \in L^1(\mathbb{R})$. Assume that

$$\int_U f(x)dx = \int_{\overline{U}} f(x)dx$$

for every open set $U \subset \mathbb{R}$. Show that $f(x) = 0$ a.e.

Attempt one of the following two problems:

11. Let X be a countably infinite set. Show that there exists a collection $\mathcal{J}$ of subsets of X such that $\mathcal{J}$ has the cardinality of continuum and $S \cap T$ is finite for all distinct $S, T \in \mathcal{J}$.

12. Show that every infinite set can be partitioned into two subsets A, B such that $\text{card}A = \text{card}B$.