DEPARTMENT OF PURE MATHEMATICS

COMPREHENSIVE EXAMINATION IN ANALYSIS AND TOPOLOGY

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Answer all the questions to the best of your ability. The exam's duration is three hours. The marks for each question are indicated in the margins.

[8] 1. (a) What is meant by an orthonormal set in a Hilbert space?

(b) Show that every Hilbert space has a maximal orthonormal set.

[10] 2. (a) Let Ω be a well-ordered set with a maximal element w. The order topology on Ω is given by the sub-base consisting of all segments of the form $I_a = \{x \in \Omega : x < a\}$ as well as the form $F_a = \{x \in \Omega : x > a\}$. Prove that Ω is compact with the order topology.

(b) The Cantor set C consists of all numbers x in the interval [0,1] that have a representation $x = \sum_{n=1}^{\infty} a_n / 3^n$ with $a_n = 0$ or 2.

Let $X = \prod_{n=1}^{\infty} \{0,2\}$ be the countable Cartesian product of the discrete space $\{0,2\}$. If X has the product topology, show that X is homeomorphic to C.

[10] 3. (a) What does it mean to say that an open subset of $\mathbb{C}$ is simply connected?

(b) State the Riemann mapping theorem.

(c) Exhibit a conformal mapping of the cut plane $\mathbb{C}^{\#} = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ onto the disk $D = \{z \in \mathbb{C} : |z| < 1\}$.

[10] 4. (a) There are several "open mapping" theorems and related "inverse function" theorems in topology and analysis. State some such theorems in the context of

(i) analytic function theory.
(ii) linear transformations between Banach spaces.
(iii) continuously differentiable maps from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$.
(iv) compact Hausdorff spaces.

(b) Give an example of a continuous linear bijection between normed linear spaces, for which the inverse is not continuous.

[15] 5. (a) Let X be a metric space and let $k : X \times [0,1] \to \mathbb{R}$ be a function such that

- for each $x \in X$, $t \mapsto k(x,t)$ is measurable.
- for each $t \in [0,1]$, $x \mapsto k(x,t)$ is continuous.
- there is an integrable function $g \in \mathcal{L}^1[0,1]$ such that $|k(x,t)| \le g(t)$ for all $x \in X$, $t \in [0,1]$.

Prove that the function $f : X \to \mathbb{R}$ given by $f(x) = \int_0^1 k(x,t)\,dt$ is continuous.

(b) By considering the closed unit ball B of the normed linear space $C[0,1]$ (continuous real valued functions defined on the interval $[0,1]$ with the uniform norm), show that there is a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \int_0^1 \frac{\sin t}{2e^x + f(t)^2}\,dt \quad \text{for all } x \in \mathbb{R}.$$

(c) Show that the solution f in (b) is unique even in the class of all measurable functions.

[10] 6. (a) Suppose $f : [0,1] \to \mathbb{C}$ is in $\mathcal{L}^1[0,1]$ and $Lf : \mathbb{C} \to \mathbb{C}$ is defined by

$$Lf(z) = \int_0^1 f(t)\, e^{zt}\,dt \quad \text{for } z \in \mathbb{C}.$$

Prove that Lf is an entire function.

(b) Prove that L is a one to one mapping of $\mathcal{L}^1[0,1]$ into the space E of entire functions.

(c) Prove that the space E, the complex space $C[0,1]$ and $\mathcal{L}^1[0,1]$ all have the same cardinality.

[10] 7. Use the residue theorem to compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{1+t^2}\,dt$ for each nonnegative real number λ.

[7] 8. Show that the complex polynomial $p(z) = 2z^6 + 6i\,z^2 + 3$ has exactly four zeroes (counting multiplicities) in the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$.

[10] 9. (a) If X is a Banach space and the series $\sum_{k\ge 0} x_k$ in X is such that $\sum_{k\ge 0} \|x_k\|$ converges, prove that $\sum_{k\ge 0} x_k$ converges.

(b) Let $z_1, z_2, z_3, \ldots$ be a sequence of complex numbers, and $r = \limsup_{k \to \infty} |z_k|^{1/k}$. Prove that the series $z_1 + z_2 + z_3 + \ldots$ converges if $r < 1$, and diverges if $r > 1$.

[10] 10. (a) Let λ denote Lebesgue measure on $\mathbb{R}$. For every subset A of $\mathbb{R}$ let $1_A : \mathbb{R} \to \mathbb{R}$ be the function which takes value 1 on A and 0 on $\mathbb{R} \setminus A$. If A is bounded and Lebesgue measurable and $\varepsilon > 0$, prove that there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ of compact support which agrees with 1_A except on a set of measure less than ε, i.e. $\lambda\{t \in \mathbb{R} : \varphi(t) \neq 1_A(t)\} < \varepsilon$.

(b) Given that the simple functions form a dense set in $\mathcal{L}^1(\mathbb{R})$, explain how part (a) can be used to show that the space $C_c(\mathbb{R})$ of continuous functions of compact support is dense in $\mathcal{L}^1(\mathbb{R})$.