Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

May 24, 1996

J. Baker & C.T. Ng

Time: 3 hours

INSTRUCTIONS: Attempt **eight** questions including at least **two** from each of the three parts.

Part A: (attempt at least two)

A.1 State and prove the Bernstein Theorem on cardinality.

A.2 Suppose that S_1, S_2 and S_3 are pairwise disjoint, nonempty closed subsets of a normal topological space X and suppose y_1, y_2 and y_3 are real numbers. Prove that there is a continuous function $f : X \to \mathbb{R}$ such that
$f(x) = y_k$ for all $x \in S_k \ , \ 1 \leq k \leq 3$.

A.3 (a) Give the definition of an ordinal number.

(b) Prove that every infinite set can be partitioned as the disjoint union of three subsets all of the same cardinality.

Part B: (attempt at least two)

B.1 Find $\displaystyle\int_0^\infty \frac{1}{a^4 + x^4}\, dx \qquad \text{for} \qquad 0 < a \in \mathbb{R}.$

B.2 (a) (i) Define what it means to say that an open subset U of $\mathbb{C}$ is simply connected.

(ii) State the Riemann mapping theorem.

(b) Exhibit a conformal mapping of

$$\begin{aligned} &Q := \{x + iy \ : \ x, y \in \mathbb{R} \ , x > 0 \ , \ |y| < x\} \ \text{ onto} \\ &D := \{z \in \mathbb{C} \ : \ |z| < 1\} \ . \end{aligned}$$

B.3 (i) Prove that for every $a \in \mathbb{R} \setminus \mathbb{Z}$,

$$\cos ax = \sin \pi a \left[\frac{1}{\pi a} + \frac{2a}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{a^2 - k^2} \right] \quad \text{for} \quad -\pi \leq x \leq \pi \ .$$

(ii) Prove that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2} \quad \text{for all} \quad z \in \mathbb{C}\backslash\mathbb{Z}. \ .$$

(You may use (i) even if you haven't proved it.)

Part C: (attempt at least two)

C.1 (i) State and prove Banach's fixed point theorem (also known as the contraction mapping principle).

(ii) Prove that there is a unique bounded function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(*) \qquad 2f(x) - f(2x) + \sin x = 0 \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Also prove that f is continuous.

C.2 Suppose that (X, d) is a metric space, $K : X \times \mathbb{R} \to \mathbb{R}$, K is continuous and bounded and suppose that $f : [0, 1] \to \mathbb{R}$ is Borel measurable.

(i) Prove that for each $x \in X$ the map $t \to K(x, f(t))$ belongs to $L^1[0, 1]$.

(ii) If $g(x) = \int_0^1 K(x, f(t)) \, dt$ for $x \in X$, prove that g is continuous.

C.3 Suppose that $f : [0, 1] \to \mathbb{R}$.

(i) Prove that f is continuous if and only if its graph $G := \{(x, f(x)) \ : \ 0 \le x \le 1\}$ is a compact subset of $\mathbb{R}^2$.

(ii) Is (i) true if "compact" is replaced by "connected"? Briefly justify your answer.

C.4 Let $C = \{(x_1, x_2, \ldots) \in l^2 \ : \ |x_k| \le \frac{1}{k}$ for all $k \in \mathbb{N}\}$ - the **Hilbert cube**. Prove that C is a compact subspace of l^2.

C.5 Suppose that $G \ : \ [0, 1] \times \mathbb{R} \to \mathbb{R}$, G is continuous and $1 \le c \in \mathbb{R}$. Let K be the set of all continuous functions $f \ : \ [0, 1] \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$ and

$$|f(s) - f(t)| \le c \, |s - t| \quad \text{for all} \quad s, t \in [0, 1] \ .$$

Prove that $\left\{ \int_0^1 G(t, f(t)) \, dt \ : \ f \in K \right\}$ has a least member.