Department of Pure Mathematics

Analysis and Topology Comprehensive Examination

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3 hours

Answer all of questions 1, 2, 3, 4, plus one of 5 or 6 and one of 7 or 8, for a total of 80 marks. The marking scheme is indicated in the margins.

1. Let E be a set of real numbers.

(2) a) Define the Lebesgue outer measure $\mu^* E$.

(2) b) What does it mean for E to be Lebesgue measurable?

(2) c) If $\mu^* E = 0$, show that E is Lebesgue measurable.

(4) d) Show that if E is bounded and measurable and if $\epsilon > 0$, then there is an open set V and a closed set F such that $F \subseteq E \subseteq V$ and $\mu(V \setminus F) < \epsilon$.

(5) e) Demonstrate the existence of a non-measurable set. Indicate in your argument where the axiom of choice is used.

2. Let $C[0,1]$ be the space of continuous complex valued functions on the interval $[0,1]$, normed according to

$$||f|| = \sup_{t \in [0,1]} |f(t)|.$$

(5) a) Prove that $C[0,1]$ is a complete metric space.

(3) b) Show that the family of functions f of the form

$$f(x) = \sum_{k=0}^{n} a_k e^{kx} \qquad a_k \in \mathbb{C}, n \in \mathbb{N},$$

is dense in $C[0,1]$.

c) Let $\varphi : C[0,1] \to C[0,1]$ be defined by

$$(\varphi f)(x) = 3 + \int_0^x tf(t)dt, \qquad \text{for} \quad f \in C[0,1], x \in [0,1].$$

Show that φ is a contraction mapping.

Find f in $C[0,1]$ such that

$$f(x) = 3 + \int_0^x tf(t)dt \quad \text{for all} \quad x \in [0,1].$$

(2)3. a) A Dedekind cut is a proper non-empty subset C of $\mathbb{Q}$ such that

$$x \in C\,,\ y \leq x \Rightarrow y \in C$$

$$\forall x \in C \ \ \exists y \in C \ \text{ such that } \ x < y.$$

The set R of Dedekind cuts is ordered by inclusion. Show that this ordering is total and that any bounded family of cuts has a least upper bound.

(5) b) Prove the Schröder-Bernstein Theorem. You may use the following guidelines, if you wish.

Let U, V be sets and $\mathcal{P}(U), \mathcal{P}(V)$ their power sets. Suppose $f : U \to V, g : V \to U$ are injections. Define $\varphi : \mathcal{P}(U) \to \mathcal{P}(U)$ by $\varphi(X) = U \setminus g(V \setminus f(X))$. Take

$$D = \cup\{X \in \mathcal{P}(U) : X \subset \varphi(X)\}.$$

Prove that $D = \varphi(D)$, and $U \backslash D \subset g(V)$.

Show that $h : U \to V$ defined by

$$h(x) = \begin{cases} f(x) & x \in D \\ g^{-1}(x) & x \in U\backslash D \end{cases}$$

is a bijection.

(3) c) Find the cardinality of the set $C(\mathbb{R})$ of continuous real valued functions on $\mathbb{R}$. Justify your answer briefly.

(5)4. a) Let the complex power series $\sum_{n=0}^{\infty} a_n z^n$ define an analytic function $f(z)$ on some disk centered at the origin.

If $r > 0$ is less than the radius of convergence of this series prove directly that

$$|a_n| \leq \frac{1}{r^n} \sup_\theta |f(re^{i\theta})|.$$

(4) b) Prove that a bounded entire function must be constant.

(5) c) Prove that a polynomial of positive degree with complex coefficients has at least one complex root.

(5) d) Find all analytic bijections $f : \mathbb{C} \to \mathbb{C}$.

(4) e) Find the maximum value of $|2z^2 - 1|$ over the closed unit disk

$$\overline{D} = \{z \ \in \ \mathbb{C} : |z| \leq 1\}.$$

Answer only one of questions 5 or 6.

(8)5. Calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ using the method of contour integration and residues.

(8)6. Show that every compact Hausdorff topological space is normal.

Answer only one of questions 7 or 8.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. If $p \in \mathbb{R}$ the *saltus* of f at p is defined by

$$S(p) = \inf_{n \in \mathbf{N}} \sup \left\{ |f(x) - f(y)| : x, y \in \left(p - \frac{1}{n}, p + \frac{1}{n} \right) \right\}$$

(4) a) Prove that for every $\delta > 0$ the set $\{p : S(p) \geq \delta\}$ is closed.

(4) b) Show that the set of discontinuities of f is an F_σ set. (i.e. a countable union of closed sets.)

(4) c) Using the Baire category theorem, show that the set of irrational numbers cannot be precisely the set of discontinuities of f.

8. Let $\mathcal{L}^1(\mathbb{R})$ be the space of integrable complex valued functions on $\mathbb{R}$ with the norm

$$||f|| = \int_{-\infty}^{\infty} |f(t)| dt \quad , \qquad f \in \mathcal{L}^1(\mathbb{R})$$

If $f \in \mathcal{L}^1(\mathbb{R})$ let

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-ixt} f(x) dx \quad , \qquad t \in \mathbb{R}.$$

(4) (a) Prove that $\hat{f}$ is a continuous bounded function.

(5) (b) Using the fact that step functions of compact support are dense in $\mathcal{L}^1(\mathbb{R})$, prove that $\hat{f}(t) \to 0$ as $|t| \to \infty$ for any f in $\mathcal{L}^1(\mathbb{R})$.

(3) (c) If $f, g \in \mathcal{L}^1(\mathbb{R})$ and $h(y) = \int_{-\infty}^{\infty} f(y-x)g(x)dx$, show that $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$ for all $t \in \mathbb{R}$. The measurability of the map $(x, y) \mapsto f(y-x)g(x)$ may be assumed.