UNIVERSITY OF WATERLOO
Department of Pure Mathematics

Comprehensive Examination in Analysis

May, 1990

(10) 1. a) State Zorn's lemma.

b) Use Zorn's lemma to prove that any set can be well-ordered.

c) Let A be a well-ordered set. Prove that if $f: A \to A$ is an order-preserving bijection, then f is the identity map.

(10) 2. a) Show that a filter $\mathcal{F}$ of a set X is a maximal (i.e. an ultrafilter) if and only if for every subset $A \subseteq X$ either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.

b) Define a compact topological space (using filters or otherwise).

c) Prove Tychonoff's theorem, that a product of compact topological spaces is compact.

(5) 3. Prove that a metric space is separable if and only if its topology has a countable base, i.e. X is second countable.

(15) 4. a) Suppose $f: [a,b] \to [0,\infty)$ is a non-negative Lebesgue measurable function and $\int_a^b f = 0$. Prove that $f = 0$ almost everywhere.

b) Find a sequence of measurable functions $f_n: [0,1] \to [0,\infty)$ such that $f_{n+1} \le f_n$, $f_n \to 0$ pointwise, but $\int_0^1 f_n \not\to 0$.

c) Explain why a measurable set $A \subseteq [0,1]$ such that, $m(A \cap J) = \frac{1}{2} m(J)$ for every subinterval $J \subseteq [0,1]$, cannot exist. Here m is Lebesgue measure.

d) If $f_n: [0,1] \to [0,\infty)$ is a sequence of measurable functions such that $\int_0^1 f_n \to 0$, does $f_n \to 0$ almost everywhere on $[0,1]$?

e) Let $f: [0,1] \to \mathbb{R}$ be the Cantor ternary function. Calculate $\int_0^1 f$.

(12) 5. (a) Let $g_n: [a,b] \to [0,\infty)$ be a sequence of continuous functions, which decreases pointwise to 0, i.e. $\forall x \in [a,b]$ $g_{n+1}(x) \le g_n(x)$ and $g_n(x) \to 0$ as $n \to \infty$. Prove that $g_n \to 0$ uniformly.

(b) Let $p_n: [0,1] \to \mathbb{R}$ be the sequence of polynomials defined recursively for each x in $[0,1]$ by $p_1(x) = 0$,

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x)).$$

(i) Prove inductively that for all n and x, $p_n(x) \le \sqrt{x}$. Hint: Consider $\sqrt{x} - p_{n+1}(x)$.

(ii) Explain why p_n is an increasing sequence of polynomials.

(iii) Why does $\lim_{n\to\infty} p_n(x) = f(x)$ exist? Find $f(x)$.

(iv) Prove that $p_n \to f$ uniformly.

(14) 6. a) Prove that every continuous function $f: [a,b] \to \mathbb{R}$ is the uniform limit of a sequence of step functions. Recall that a step function takes only finitely many values at a finite number of intervals.

b) Let $f: [a,b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n \to \infty} \int_a^b f(x) \cos nx \, dx = 0.$$

c) State the Stone-Weierstrass theorem for the case of complex valued functions defined on a compact set.

d) Suppose $f: [-\pi, \pi] \to \mathbb{R}$ is continous, $f(-\pi) = f(\pi)$ and for all n in $\mathbb{Z}$, $\int_{-\pi}^{\pi} f(x) e^{inx} dx = 0$. Prove $f = 0$.

(6) 7. a) Let U be an open set in the complex plane $\mathbb{C}$ and $p \in U$. Let $f: U \setminus \{p\} \to \mathbb{C}$ be analytic. Define what it means to say that p is a removable singularity of f, a pole of f, an essential singularity of f.

b) Let f be analytic on the punctured disk $\{z : 0 < |z| < 2\}$. Suppose that $\int_{|z|=1} z^n f(z) dz = 0$ for $n = 0,1,2,\ldots$. What type of singularity does f have at 0? Explain.

(8) 8. Using the method of residues compute $\int_0^\infty \frac{\sin x}{x} dx$.

(10) 9. a) State the maximum modulus principle.

b) Prove Schwarz's lemma. Namely, if f is analytic in the disk $D = \{z : |z| \leq 1\}$, $f(0) = 0$, and $|f(z)| < 1$ for $z \in D$, then $|f(z)| \leq |z|$ for $z \in D$. Also, if $|f(z_0)| = |z_0|$ for some non-zero point z_0, then $f(z) = \lambda z$ for some λ on the unit circle.

c) Describe all the conformal equivalences $f: D \to D$, (i.e. f is an analytic bijection with analytic inverse) such that $f(0) = 0$.

(10) 10. a) Prove that the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ is homeomorphic to the plane $\mathbb{C}$.

b) Prove that D is not conformally equivalent to $\mathbb{C}$.

c) Under the conformal mapping given by $w = \frac{z}{z-1}$ determine the images of the half-plane $\{z : \mathrm{Re}(z) \geq 0\}$, and of the vertical strip $\{z : 0 < \mathrm{Re}(z) < 1\}$.