DEPARTMENT OF PURE MATHEMATICS

COMPREHENSIVE EXAMINATION IN ANALYSIS AND TOPOLOGY

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Examiners: J. Baker & K. Hare

Time: 3 HOURS
INSTRUCTIONS: ANSWER
ALL OF QUESTIONS 1 THROUGH 6,
ONE OF EITHER 7 OR 8
AND
ONE OF EITHER 9 OR 10.

1. (a) State the contraction mapping principle which is also called Banach's fixed point theorem.

   (b) Suppose $\alpha, \beta : [0,1] \to \mathbf{R}$ and $\gamma : [0,1] \to [0,1]$ are given continuous functions and $|\beta(x)| < 1$ for all $x \in [0,1]$. Prove that there is a unique continuous function $f : [0,1] \to \mathbf{R}$ such that $f(x) = \alpha(x) + \beta(x) f(\gamma(x))$ for all $x \in [0,1]$.

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable functions from $[0,1]$ to $\mathbf{R}$, $M > 0$ and, for all $n \in \mathbf{N}$, $|f_n(t)| \leq M$ for *a.e.* $t \in [0,1]$. Let $g_n(x) = \int_0^x f_n(t)dt$ for $0 \leq x \leq 1$ and $n \in \mathbf{N}$.

   (a) Prove that g_n is continuous for every $n \in \mathbf{N}$.

   (b) Prove that some subsequence of $\{g_n\}_{n=1}^{\infty}$ is uniformly convergent.

3. (a) Prove that an open, connected subset of a real normed linear space is path connected.

   (b) What does it mean to say that an open subset of $\mathbf{C}$ is simply connected?

   (c) State the Riemann mapping theorem.

   (d) Find an analytic bijection of

$$U := \{z \in \mathbf{C} : Re z > 0 \text{ and } Im z > 0\} \text{ onto}$$

$$D := \{z \in \mathbf{C} : |z| < 1\}.$$

   (e) Prove that there is no analytic bijection of D onto $\mathbf{C}$.

4. Suppose f is an analytic function on $D := \{z \in \mathbf{C} : |z| < 1\}$ with $|f(z)| \leq 2\ |z|^{\frac{3}{2}}$ for all $z \in D$. Show that actually $|f(z)| \leq 2\ |z|^2$ for all $z \in D$.

5. (a) Suppose $0 \leq \alpha < 1 < \beta$, $A = \{z \in \mathbf{C} : \alpha < |z| < \beta\}$, $f : A \to \mathbf{C}$ is analytic and $g(t) = f(e^{it})$ for $t \in \mathbf{R}$. Explain how to obtain the Fourier series of g from the Laurent expansion of f. What can be said concerning the convergence of the Fourier series of g?

(b) Find the Fourier series of g if

$$g(t) = \frac{3}{5 - 4\ cost} \quad \text{for } t \in \mathbf{R}.$$

6. Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue integrable on $\mathbf{R}$.
For $x \in \mathbf{R}$ let $f_x(t) = f(x+t)$ for $t \in \mathbf{R}$. Prove that $\lim_{x \to 0} \|f_x - f\|_1 = 0$. Hint: Consider that case in which f is continuous and has compact support.

Answer <u>one</u> of the following two questions.

7. (a) State Fatou's lemma.
   (b) Show that, for each $n \in \mathbf{R}$, $\{f \in L^1[0,1] : \int_0^1 |f(x)|^2 dx \leq n\}$ is closed and has empty interior in $L^1[0,1]$.
   (c) Prove that $L^2[0,1] \subseteq L^1[0,1]$.
   (d) (i) Define "first category".
   (ii) Prove that $L^2[0,1]$ is of first category in $L^1[0,1]$.

8. Show that

$$\int_{-\infty}^{\infty} e^{-t^2/2} \quad e^{i\lambda t} \quad dt = \sqrt{2\pi}\ e^{-\lambda^2/2}$$

for $\lambda > 0$ by considering

$$\int_{\gamma_R} e^{-z^2/2} \quad e^{i\lambda z}\ dz$$

where γ_R is the positively oriented rectangle with vertices $R, R + i\lambda$, $-R + i\lambda$ and -R. $Hint: \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$.

Answer <u>one</u> of the following two questions.

9. (a) Use the axiom of choice (or an equivalent statement) to prove that every vector space has a basis (in the algebraic sense).
   (b) Give an example (without proof) of a compact Hausdorff space which is not metrizable.

10. Prove that the closed unit ball of a Hilbert space, H, is compact if and only if H is finite dimensional.