Department of Pure Mathematics

Analysis Comprehensive Examination
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Attempt 8 questions, including at least two from each part.

Part A

1. (a) Prove the set $\{f \in C[0,1] : \|f\| \leq 1\}$ is closed and bounded but not compact.

   (b) Prove that the set $\{f \in C[0,1] : |f(x) - f(y)| \leq \sqrt{|x-y|} \quad \text{and} \quad f(0) = 0\}$ is compact.

2. Let $2^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n = 0, 1\}$ be given the usual product topology.

   (a) Prove that $2^{\mathbb{N}}$ is metrizable and give a formula for an appropriate metric.

   (b) Let $T : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be given by $T(s_1, s_2, s_3 \ldots) = (s_2, s_3, \ldots)$. Prove T is uniformly continuous.

3. Evaluate $\displaystyle \lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$.

4. Let the real–valued function f on $[0,1]$ have the following two properties:

   (i) if $[a,b] \subseteq [0,1]$, then $f([a,b])$ contains the interval with endpoints $f(a), f(b)$ (i.e., f has the intermediate value property).

   (ii) for every $c \in \mathbb{R}$, the set $f^{-1}(\{c\})$ is closed.

   Prove that f is continuous.

Part B

1. (a) Suppose $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and $f : \mathbb{D} \to \mathbb{C}$ is analytic. Suppose there is a constant M such that whenever $\{z_n\} \subseteq \mathbb{D}$ has $\lim z_n \in$ boundary $\mathbb{D}$, then $\limsup |f(z_n)| \leq M$. Prove $|f(z)| \leq M \ \ \forall\, z \in \mathbb{D}$.

   (b) Suppose f and g are entire functions and $|f(z)| \leq |g(z)|\ \forall z$. Prove there is a constant c such that $f(z) = cg(z)\ \forall z$.

2. Evaluate
$$\int_0^{\infty} \frac{\sin x}{x(1+x^2)}\, dx$$

3. Suppose that f is an analytic function on the upper half-plane which satisfies $|f(z)| \leq 1$ for all z and $f(i) = 0$. Prove $|f(2i)| \leq \frac{1}{3}$.

Part C

1. Prove there does not exist a continuous function $f : [0,1] \to \mathbb{C}$ such that

   (i) $\displaystyle\int_0^1 x f(x) dx = 1$ and

   (ii) $\displaystyle\int_0^1 x^n f(x) dx = 0 \quad \text{for} \quad n = 0, 2, 3, 4, \ldots$

2. Let $C \subseteq \mathbb{R}$ be compact and assume $m(C) > 0$. (Here m denotes Lebesgue measure on $\mathbb{R}$.)

   (a) Using the definition of m explain why there is an open set $U \supseteq C$ such that $m(U) < 2m(C)$.

   (b) Fix such an open set U. Let $\delta = \inf\{|x - y| \,:\, x \in C, y \in U^c\}$. Prove $(-\delta, \delta) \subseteq C - C \equiv \{c_1 - c_2 : c_1, c_2 \in C\}$.

   (c) Prove that if $E \subset \mathbb{R}$ is measurable and $m(E) > 0$ then $E - E$ has non-empty interior.

3. Let $g \in L^2[0, 2\pi]$. Recall that for $n \in \mathbb{Z}$

$$\hat{g}(n) \equiv \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx}\, dx \,.$$

   Define $g_N(t) \equiv g(Nt)$ where g has been extended by 2π–periodicity to all of $\mathbb{R}$.

   (a) Show that $\hat{g}_N(jN) = \hat{g}(j)$ for $j \in \mathbb{Z}$ and $\hat{g}_N(k) = 0$ if $k \in \mathbb{Z}$ and k is not divisible by N.

   (b) Let $f \in L^2[0, 2\pi]$. Prove

$$\lim_{N \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g}_N(t)\, dt = \hat{f}(0) \overline{\hat{g}(0)} \,.$$

4. Let f_n be a sequence of real–valued measurable functions on $[0,1]$ and α_n a sequence of positive numbers. Suppose

$$\sum_{n=1}^{\infty} m\{x \in [0,1] \,:\, |f_n(x)| > \alpha_n\} < \infty \,.$$

   (Here m denotes Lebesgue measure on $[0,1]$.)

   Prove $-1 \leq \liminf \dfrac{f_n(x)}{\alpha_n} \leq \limsup \dfrac{f_n(x)}{\alpha_n} \leq 1 \quad$ a.e.