Department of Pure Mathematics
Analysis and Topology Comprehensive Examination

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Time: 3 hours

Marks, out of 100, are indicated in square brackets in the left margin.

The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all integers, rational numbers, real numbers and complex numbers respectively.

[5] 1. Exhibit a bijection of $[0,1]$ onto $[0,1)$. Could such a function be continuous?

[8] 2. Suppose that X and Y are real vector spaces, X_0 is a linear subspace of X and $L_0 : X_0 \to Y$ is linear. Use Zorn's lemma to prove that there exists a linear map $L : X \to Y$ such that

$$L_0(x) = L(x) \quad \text{for all} \quad x \in X_0.$$

[12] 3. (i) What does it mean to say that a topological space is normal?

(ii) Prove that every compact Hausdorff space is normal.

(iii) State Urysohn's lemma.

[12] 4. (i) Prove that $L^2[0,1] \subseteq L^1[0,1]$ but $L^2(\mathbb{R}) \not\subseteq \mathrm{L}^1(\mathbb{R})$.

(ii) Does there exists $A > 0$ such that

$$\left(\int_0^1 |f(t)|^2 \, dt\right)^{\frac{1}{2}} \leq A \int_0^1 |f(t)| \, dt$$

for all $f \in L^2[0,1]$?

(iii) Does there exist $B > 0$ such that

$$\int_0^1 |f(t)| \, dt \leq B \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{for all} \quad f \in L^2[0,1]?$$

[8] 5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and f is differentiable then f' is Borel measurable.

[10] 6. Suppose that (X, d) is a metric space. A sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be a **fast Cauchy sequence** provided $\sum_{n \geq 1} d(x_n, x_{n-1})$ is convergent.

(i) Prove that every Cauchy sequence in X has a fast Cauchy subsequence.

(ii) Prove that (X, d) is complete if and only if every fast Cauchy sequence in X is convergent.

[8] 7. Suppose that A and B are nonempty subsets of $\mathbb{R}$ and $C = \{x+y : x \in A \text{ and } y \in B\}$.

(i) Prove that if either A or B is open then C is open.

(ii) Prove that if both A and B are compact then C is compact.

[12] 8. Suppose that w_1 and w_2 are nonzero complex numbers.

(i) Prove that if f is an entire function such that

$$(*) \qquad f(z) = f(z+w_1) = f(z+w_2) \ \text{ for all } \ z \in \mathbb{C}$$

and if $w_1/w_2 \notin \mathbb{Q}$ then f is constant.

(ii) Prove that if $w_1/w_2 \in \mathbb{Q}$ there exists a nonconstant entire function f satisfying (*).

[15] 9. Let $D = \{z \in \mathbb{C} : |z| < 1\}$.

By a **region** we mean a nonempty open connected subset of $\mathbb{C}$.

(i) What does it mean to say that a region is simply connected?

(ii) State the Riemann Mapping Theorem.

(iii) Find a conformal mapping, f, of $W := \{x + iy \in \mathbb{C} : x > 0 \text{ and } |y| < x\}$ onto D.

(iv) Prove that $\mathbb{C}$ is homeomorphic to D.

(v) Prove that if U and V are simply connected regions then U is homeomorphic to V.

[10] 10. Find $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{(1+t^2)^2}\, dt \qquad \text{for } 0 < \lambda \in \mathbb{R}.$