Department of Pure Mathematics
Analysis and Topology Comprehensive Examination

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Time: 3 hours

In each question, be careful to clarify all details concerning interchange of limit operations, the appropriate definitions of functions when there may be ambiguity, et cetera.

Do ALL of the following questions:

1. Suppose that U is a simply connected region in the complex plane, and let f be an analytic function defined on U.

   a) Show that there is an analytic function g on U such that
   $$g'(z) = f(z) \quad \text{for all} \quad z \in U \ .$$

   b) Show that if $f(z) \neq 0$ for all $z \in U$, then there is an analytic function h on U such that
   $$f(z) = e^{h(z)} \quad \text{for all } z \in U \ .$$

   c) Show that both of these results are false for
   $$U = \{z \in \mathbb{C} : z \neq 0\} \ .$$

2. a) Show that the set of points of discontinuity of a monotonoic increasing function on $\mathbb{R}$ is at most countable.

   b) Show that every continuous function from $[2, 3]$ into itself has a fixed point.

   c) Suppose that a real valued function f on $\mathbb{R}$ satisfies
   $$\lim_{h\to 0}\ [f(x+h) - f(x-h)] = 0, \ \text{for all } x \in \mathbb{R} \ .$$
   Is f necessarily continuous?

3. Let $f(x)$ be a twice continuously differentiable (C^2) 2π-periodic function on the real line with Fourier series $f \sim \displaystyle\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.

   a) Prove that the Fourier series converges absolutely and uniformly to a continuous function g.

   b) Compute the Fourier series of g.

   c) Hence show that $g = f$.

4. TRUE or FALSE: give a proof or counterexample. The functions f_n lie in $L^1[0,1]$ for $n \geq 0$.

   a) $f_n \geq 0$ and $\lim_{n\to\infty} f_n(x) = 0$ for every $x,\ 0 \leq x \leq 1$. Then
$$\lim_{n\to\infty} \int f_n = 0\ .$$

   b) $f_n \geq 0$. Then
$$\int \sum_{n\geq 0} f_n = \sum_{n\geq 0} \int f_n\ .$$

   c) $f_n \geq 0, f_{n+1} \leq f_n$ for for every $x,\ n \geq 0$ and $\lim_{n\to\infty} f_n(x) = 0$ for $0 \leq x \leq 1$. Then
$$\int \sum_{n>0} (-1)^n f_n = \sum_{n\geq 0} \int (-1)^n f_n\ .$$

5. Let $\mathbb{R}^{\aleph_0}$ be given the *box topology* determined by a base of open sets of the form
$$U = U_1 \times U_2 \times \ldots = \prod_{i\geq 1} U_i$$
where U_i are open subsets of $\mathbb{R}$ in the usual topology.

   a) What is the relationship of this topology to the product topology?
   b) Let $0 = (0,0,\ldots)$. Find a net x_α such that $x_\alpha = (x_{\alpha,k})_{k\geq 1}$, such that $x_{\alpha,k} \neq 0$ for every α, k and x_α converges to 0.
   c) Show that this net cannot be chosen to be a sequence.

6. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space.

   a) Prove that H has an *algebraic* basis (that is, a linearly independent spanning subset) as a vector space.
   b) What is the cardinality of this basis?
   c) What are the possible cardinalities for the algebraic basis of a *closed* subspace of $\mathcal{H}$?

Do TWO of the following questions:

7. Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}\,dx$.

8. a) How many roots does the polynomial $p(z) = 2z^5 - 6iz^2 + iz - 1$ have in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

   b) Consider two polynomials $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ and $q(z) = z^m + b_{m-1}z^{m-1} + \ldots + b_0$. Suppose that

$$\{z \in \mathbb{C} : |p(z)| > 1\} \subset \{z \in \mathbb{C} : |q(z)| > 1\} .$$

   Show that $p^m = q^n$.
   Hint: Consider $h(z) = \dfrac{p^m}{q^n}$ in a neighbourhood of ∞.

9. Let $X = \{0,1\}^{\aleph_0}$ and $Y = \{0,1,2\}^{\aleph_0}$ with the product topologies. Prove that X and Y are homeomorphic.

10. Let μ be a signed measure on a measure space (X, Σ) such that

$$\sup_{A \in \Sigma} \mu(A) = L < \infty .$$

   a) If $A_n \in \Sigma$ and $\mu(A_n) > L - \varepsilon_n$, show that for each $k \geq 1$

$$\mu\left(\bigcap_{n \geq k} A_n\right) > L - \sum_{n \geq k} \varepsilon_n.$$

   b) Show that there is a set $A_0 \in \Sigma$ such that $\mu(A_0) = L$.