Pure Mathematics Complex Analysis Qualifying Examination
University of Waterloo
September 15, 2022

Instructions

1. Print your name and UWaterloo ID number at the top of this page, and on no other page.

2. Check for questions on both sides of each page.

3. Answer the questions in the spaces provided. If you require additional space to answer a question, please use one of the overflow pages, and refer the grader to the overflow page from the original page by giving its page number.

4. Do not write on the Crowdmark QR code at the top of each page.

5. Use a dark pencil or pen for your work.

6. All questions are equally weighted.

1. Let $a \in \mathbb{C}$ be such that $0 < |a| < 1$ and $n \in \mathbb{N}$. Show that $(z-1)^n e^z - a = 0$ has exactly n simple roots in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

2. Let $f : \Omega \to \Omega$ be a holomorhic map where $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. Suppose that f is not the identity map on Ω. Show that f can have at most one fixed point.
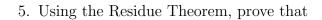
Extra page for answers. Please specify the question number here and the use of this page on the question page.

3. Show that an entire function f satisfying $|f(z)| \leq \sqrt{2 + |z|}$ for all $z \in \mathbb{C}$ is constant.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant non-vanishing entire function. Let $\Omega = \{z \in \mathbb{C} : |f(z)| < 1\}$. Show that Ω is unbounded.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

5. Using the Residue Theorem, prove that

$$\int_0^{2\pi} \sin^{2k}(\theta) \, d\theta = \frac{\pi(2k)!}{2^{2k-1}(k!)^2} \qquad (k = 0, 1, 2, \ldots).$$

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.

Extra page for answers. Please specify the question number here and the use of this page on the question page.